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Perturbed geodesics on the moduli space of flat connections and Yang–Mills theory

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Abstract If we consider the moduli space of flat connections of a non trivial principal SO(3)-bundle over a surface, then we can define a map from the set of perturbed closed geodesics, below a given energy level, into families of perturbed Yang–Mills connections depending on a parameter ε. In this paper we show that this map is a bijection and maps perturbed geodesics into perturbed Yang–Mills connections with the same Morse index.

Keywords Adiabatic process · Flat connection · Geodesic · Moduli space · Yang–Mills equation

Mathematics Subject Classification 53C22 · 53C07 · 53D20 · 35J60

1 Introduction

The moduli space of flat connections for a principal bundle over a surface Σ with genus g is an infinite dimensional analogue of a symplectic reduction and was investigated for the first time in 1983 by Atiyah and Bott (cf. [1]) who showed that, on this particular moduli space, one can define an almost complex structure induced by the Hodge-*-operator acting on the 1-forms over Σ and hence induced by its conformal structure; with the almost complex structure and the inner product on the 1-forms one can also obtain a symplectic form. Furthermore, if we choose a principal non trivial SO(3)-bundle, then the moduli space $\mathcal{M}^g(P)$, defined as the quotient between the space of the flat connections $\mathcal{A}_0(P) \subset \mathcal{A}(P)$ and the

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identity component of the gauge group $\mathcal{G}_0(P)$, is a smooth compact symplectic manifold of dimension $6g-6$ (cf. [2]). In the nineties some aspects of the topology of $\mathcal{M}^g(P)$ were investigated by Dostoglou and Salamon (cf. [3]), who proved an isomorphism between the symplectic and the instanton Floer homology related to this moduli space, and in the work of Hong (cf. [5]). Hong took an oriented compact manifold B with a Riemannian metric g_B and a harmonic map $\phi : B \to \mathcal{M}^g(P)$ and he showed that if the Jacobi operator of ϕ is invertible, then there exist a constant ε_0 and, for $0 < \varepsilon < \varepsilon_0$, a family A^ε of Yang–Mills connections of the principal SO(3)-bundles $P \times B \to \Sigma \times B$, where the base manifold has a partial rescaled metric $g_\Sigma \oplus \frac{1}{\varepsilon^2} g_B$, which converges to the connection that generates ϕ. In this paper we choose $B = S^1$ and a slightly different rescaling of the metric and we extend the results of Hong; more precisely the setting is the following one.

On the one hand, we consider the loop space on $\mathcal{M}^g(P)$ and its elements can be seen as connections $A(t) + \Psi(t)dt$ on the manifold $\Sigma \times S^1$, where $A(t) \in \mathcal{A}_0(P)$ and $\Psi(t)$ is a 0-form in $\Omega^0(\Sigma, \mathfrak{g}_P)$, satisfying the condition $d_A^*(\partial_t A - d_A\Psi) = 0$. The 1-form $\partial_t A - d_A\Psi$ corresponds to the speed vector of our loop and thus the perturbed energy functional is

$$E^H(A) = \frac{1}{2}\int_0^1 \left(\|\partial_t A - d_A\Psi\|^2_{L^2(\Sigma)} - H_t(A)\right) dt \tag{1}$$

where $H_t : \mathcal{A}(P) \to \mathbb{R}$ is a generic equivariant Hamiltonian map which is introduced in order to obtain an invertible second variational form. On the other hand, we can take the 3-manifold $\Sigma \times S^1$ with the metric $\varepsilon^2 g_\Sigma \oplus g_{S^1}$ for a positive parameter ε and consider the principal SO(3)-bundle $P \times S^1 \to \Sigma \times S^1$. In this case, for a connection $\Xi = A + \Psi\, dt \in \mathcal{A}(P \times S^1)$, where $A(t) \in \mathcal{A}(P)$, $\Psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ the curvature is $F_\Xi = F_A - (\partial_t A - d_A\Psi) \wedge dt$ and thus the perturbed Yang–Mills functional can be written as

$$\mathcal{YM}^{\varepsilon,H}(\Xi) = \frac{1}{2}\int_0^1 \left(\frac{1}{\varepsilon^2}\|F_A\|^2_{L^2(\Sigma)} + \|\partial_t A - d_A\Psi\|^2_{L^2(\Sigma)} - H_t(A)\right) dt. \tag{2}$$

Then, by a contraction argument one can define a map between the perturbed geodesics (seen as connections $A + \Phi dt$) below an energy level b, denoted by $\mathrm{Crit}^b_{E^H}$, and the set of the perturbed Yang–Mills connections $\mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$ with energy less than b provided that the parameter ε is small enough. Furthermore, this map can also be defined uniquely, it is surjective and maps perturbed geodesics to perturbed Yang–Mills connections with the same Morse index. Summarizing, in this paper, we prove the following theorem.

Theorem 1 *We assume that the Jacobi operators of all the perturbed geodesics on $\mathcal{M}^g(P)$ are invertible and we choose a regular value b of the energy E^H and $p \geq 2$. Then there are two positive constants ε_0 and c such that the following holds. For every $\varepsilon \in (0, \varepsilon_0)$ there is a unique gauge equivariant map*

$$\mathcal{T}^{\varepsilon,b} : \mathrm{Crit}^b_{E^H} \to \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$$

satisfying, for $\Xi^0 \in \mathrm{Crit}^b_{E^H}$,

$$d^{*_\varepsilon}_{\Xi^0}\left(\mathcal{T}^{\varepsilon,b}(\Xi^0) - \Xi^0\right) = 0, \quad \left\|\mathcal{T}^{\varepsilon,b}(\Xi^0) - \Xi^0\right\|_{\Xi^0,2,p,\varepsilon} \leq c\varepsilon^2. \tag{3}$$

Furthermore, this map is bijective and $\mathrm{index}_{E^H}(\Xi^0) = \mathrm{index}_{\mathcal{YM}^{\varepsilon,H}}(\mathcal{T}^{\varepsilon,b}(\Xi^0))$.

The result of Hong Hong could assume that the harmonic map ϕ has an invertible Jacobi operator because, even for an unperturbed energy functional, you can reach this condition for example for a 2-dimensional manifold B and eventually slightly perturbing the metric g_B. For $B = S^1$ the Jacobi operator of a geodesic is never invertible and for this reason we need to introduce a perturbation in our functional E^H as we will discuss in the Sect. 3. Another important point worth to be remarked is the different choice of the rescaling. On the one side, if we do not consider the Hamiltonian perturbation, both choices give the same equations for the Yang–Mills connections for $B = S^1$ and hence his methods work also in our case; we can therefore say that Hong proved the existence of the map $\mathcal{T}^{\varepsilon,b}$. However, he did not prove its uniqueness and its surjectivity. On the other side, the different choice of the metric gives two different Yang–Mills energy functionals; in fact, using the metric $g_\Sigma \oplus \frac{1}{\varepsilon^2} g_{S^1}$ one obtains the Yang–Mills energy functional $\varepsilon\mathcal{YM}^{\varepsilon,\frac{1}{\varepsilon}H}$ instead of $\mathcal{YM}^{\varepsilon,H}$ and the properties of $\mathcal{YM}^{\varepsilon,H}$ will play a major role in the proof of the surjectivity of $\mathcal{T}^{\varepsilon,b}$ and in particular, in order to obtain the a priori estimates for the curvature of the perturbed Yang–Mills connections.

Outline The following sections are of preliminary nature; in fact, first, we briefly introduce the moduli space $\mathcal{M}^g(P) := \mathcal{A}_0(P)/\mathcal{G}_0(P)$ of flat connections of a non trivial principal $\mathbf{SO}(3)$-bundle P over a surface $(\Sigma,\ g_\Sigma)$ of genus g (Sects. 2, 3). Then, on the one hand, we discuss the equations of the perturbed closed geodesics on $\mathcal{M}^g(P)$ (Sect. 4) and on the other hand, we introduce for a given $\varepsilon > 0$ the equations for the perturbed Yang–Mills connections of the principal $\mathbf{SO}(3)$-bundle $P \times S^1 \to \Sigma \times S^1$ where the metric on Σ is rescaled by a factor ε^2 (Sect. 5). Next, we also define the norm which will play a fundamental role in the proof of the Theorem 1 (Sect. 6). In the successive two sections we compute the linear (Sect. 7) and the quadratic (Sect. 8) estimates and in Sect. 9, we define the injective map $\mathcal{T}^{\varepsilon,b}$ and furthermore, we prove that this map is unique under the condition (3). In the next section, we show some a priori estimates (Sect. 10) that we need to prove the surjectivity of the map $\mathcal{T}^{\varepsilon,b}$ (Sect. 11). We prove the surjectivity of the map $\mathcal{T}^{\varepsilon,b}$ indirectly: We assume that there is a sequence of perturbed Yang–Mills connections Ξ^{ε_ν}, $\varepsilon_\nu \to 0$, which is not in the image of $\mathcal{T}^{\varepsilon_\nu,b}$, and we show that this sequence has a subsequence which converges to a geodesic Ξ^0; then using the uniqueness property of $\mathcal{T}^{\varepsilon,b}$ this subsequence turn out to be in the image of $\mathcal{T}^{\varepsilon_\nu,b}(\Xi^0)$ yielding a contradiction. In the last section, we conclude the proof of the Theorem 1 proving that $\mathcal{T}^{\varepsilon,b}$ maps perturbed geodesics to perturbed Yang–Mills connections with the same Morese index (Theorem 18); in fact the Theorem 1 follows directly from the Definition 1 of the map $\mathcal{T}^{\varepsilon,b}$, its surjectivity (Theorem 16) and the index Theorem 18.

Remark 1 We denote by $\mathcal{L}^b\mathcal{M}^g(P)$ and by $\mathcal{A}^{\varepsilon,b}(P \times S^1)$ respectively the subsets where $E^H \leq b$ and $\mathcal{YM}^{\varepsilon,H} \leq b$. Since we have a bijection between the critical points of the two functionals, we can also expect an isomorphism between the Morse homology, defined with the L^2 gradient flows, of the bounded loop space $\mathcal{L}^b\mathcal{M}^g(P)$ and that of the moduli space $\mathcal{A}^{\varepsilon,b}(P \times S^1)/\mathcal{G}_0(P \times S^1)$, as it is explained in [7]:

Theorem 2 *We assume that the energy functional E^H is Morse–Smale. For every regular value $b > 0$ of E^H there is a positive constant ε_0 such that, for $0 < \varepsilon < \varepsilon_0$, the inclusion $\mathcal{L}^b\mathcal{M}^g(P) \to \mathcal{A}^{\varepsilon,b}\left(P \times S^1\right)/\mathcal{G}_0\left(P \times S^1\right)$ induces an isomorphism*

$$HM_*\left(\mathcal{L}^b\mathcal{M}^g(P), E^H, \mathbb{Z}_2\right) \cong HM_*\left(\mathcal{A}^{\varepsilon,b}\left(P \times S^1\right)/\mathcal{G}_0\left(P \times S^1\right), \mathcal{YM}^{\varepsilon,b}, \mathbb{Z}_2\right).$$

Remark 2 The manifold $\mathcal{M}^g(P) = \mathcal{A}_0(P)/\mathcal{G}_0(P)$ can be also interpreted as a symplectic quotient defined with the moment map $\mu : \mathcal{A}(P) \to \Omega^0(\mathfrak{g}_P)$, $\mu(A) = *F_A$ and thus we can also investigate the finite dimensions analogue of the correspondence stated in the Theorem 1.

For this purpose we choose a finite dimensional symplectic manifold X and a Lie group G acting free on it; we assume in addition that a Hamiltonian action is generated by an equivariant moment map $\mu : X \to \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G, with regular value 0 and that the compatible almost complex structure J on X is G-invariant. Furthermore, we choose a time dependent and G invariant potential $V_t : X \to \mathbb{R}$. On the one side, we can study the perturbed geodesics on the symplectic quotient $\mathcal{M} := \mu^{-1}(0)/G$, that we assume compact, and hence the critical points of

$$\mathcal{E}^{\mu,V}(x,\xi) := \frac{1}{2}\int_0^1 \left(|\dot{x} + L_x\xi|^2 - V_t(x)\right) dt \tag{4}$$

for $(x,\xi) \in \mathcal{L}(\mu^{-1}(0)) \times \mathcal{L}(\mathfrak{g})$ and where $L_{x(t)}\xi(t) \in T_{x(t)}X$ denotes the fundamental vector field generated by $\xi(t) \in \mathfrak{g}$ and evaluated at $x(t)$. On the other side, we choose on the loop space of $X \times \mathfrak{g}$ the twisted energy functional

$$\mathcal{E}^{\mu,V,\varepsilon}(x,\xi) := \frac{1}{2}\int_0^1 \left(\frac{1}{\varepsilon^2}|\mu(x)|^2 + |\dot{x} + L_x\xi|^2 - V_t(x)\right) dt \tag{5}$$

for $(x,\xi) \in \mathcal{L}(X) \times \mathcal{L}(\mathfrak{g})$. This last energy functional is the analogue of the perturbed Yang–Mills energy functional $\mathcal{YM}^{\varepsilon,H}$. Also for the finite dimensional case, we can prove a bijection between the critical loops below a given energy level and for ε small enough (cf. [6]).

2 Preliminaries

In the next sections we briefly explain the setting for our results. In order to introduce the moduli space of flat connections for a non trivial principal SO(3)-bundle over a surface Σ, we first explain some facts about a principal G-bundle $\pi : P \to \Sigma$ where G is a compact Lie group with Lie algebra $\mathfrak{g}$ and P and Σ are smooth manifolds. The action of G on P defines a vertical space

$$V := \left\{\left(p,\, p\xi := \left.\frac{d}{dt}\right|_{t=0} p\exp(t\xi)\right) \,\middle|\, p \in P,\ \xi \in \mathfrak{g}\right\} \subset TP$$

in the tangent bundle and hence a choice of a connection, i.e. an equivariant function $A : TP \to \mathfrak{g}$ which satisfies

$$\begin{array}{ll} i)\ A(p,\ p\xi) = \xi & \forall p \in P,\ \forall \xi \in \mathfrak{g}, \\ ii)\ A(pg,\ vg) = g^{-1}A(p,v)g & \forall p \in P, \forall v \in T_pP, \end{array}$$

could also be seen as a choice of an equivariant horizontal distribution $H \subset TP$ which corresponds to the kernel of A and at each point $p \in P$ induces the short exact sequence

$$0 \longrightarrow H_p = \ker A(p,.) \xrightarrow{\iota} T_pP \longrightarrow V_p \longrightarrow 0,$$

where ι is the inclusion of H_p in T_pP and V_p the restriction of V at the point p. In addition, since $V_p = \ker(d\pi(p))$ and $T_pP = H_p \oplus V_p, d\pi(p)$ induces an isomorphism between H_p and $T_{\pi(p)}\Sigma$, hence the horizontal distribution is isomorph to the pullback $\pi^*T\Sigma$ and this observation implies that a vectorfield X on Σ has a unique horizontal lift $\tilde{X} \subset H$ on P such that $\tilde{X}(p) \in H_p$ and $d_p\pi(\tilde{X}(p)) = X(\pi(p))$. The set of all the connections of a principal bundle is denoted by $\mathcal{A}(P)$ and it is an affine space; in fact, for every connection $A_0 \in \mathcal{A}(P)$, $\mathcal{A}(P) = A_0 + \Omega^1_{\mathrm{Ad},H}(P,\mathfrak{g})$ where $\Omega^1_{\mathrm{Ad},H}(P,\mathfrak{g})$ denotes the set of all equivariant functions $\alpha : TP \to \mathfrak{g}$ such that $V \subset \ker \alpha$, i.e. α is horizontal. Similarly, $\Omega^k_{Ad,H}(P,\mathfrak{g})$ is the space of equivariant and horizontal k-forms, i.e for an $\omega \in \Omega^k_{Ad,H}(P,\mathfrak{g})$ we have

$$\begin{aligned} \omega(pg; v_1 g, v_2 g, \dots, v_k g) &= g^{-1}\omega(p; v_1, v_2, \dots, v_k)g, \\ \omega(p; v_1, \dots, v_k) &= 0, \quad \text{if } v_i = p\xi \text{ for an } i \in \{1,\dots,k\}, \end{aligned}$$

where $p \in P$, $g \in G$, $\xi \in \mathfrak{g}$, $v_i \in T_pP$, $1 \le i \le k$. Therefore, the equivariant and horizontal k-forms $\Omega^k_{\mathrm{Ad},H}(P,\mathfrak{g})$ correspond to the k-forms over Σ with values in the adjoint bundle, i.e. $\Omega^k_{\mathrm{Ad},H}(P,\mathfrak{g}) \cong \Omega^k(\Sigma,\mathfrak{g}_P)$, where $\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g}$ is the associated bundle defined by the equivalence classes $[pg,\xi] \equiv [p,\mathrm{Ad}_g\xi] \equiv [p, g\xi g^{-1}]$.

The Lie group $\mathcal{G}(P)$ of equivariant smooth maps $u : P \to G$ is called the gauge group of P, i.e.

$$\mathcal{G}(P) := \{u \in C^\infty(P,G) \mid u(pg) = g^{-1}u(p)g,\ \forall p \in P,\ \forall g \in G\}.$$

Since G acts on P, every element of the gauge group induces a gauge transformation of the bundle P, i.e. $\tilde{u} : P \to P;\ p \mapsto pu(p)$ which is a G-bundle isomorphism. A gauge transformation u acts on the space of connections by

$$u^*A = u^{-1}Au + u^{-1}du$$

for $A \in \mathcal{A}(P)$ and hence we can consider u as a change of coordinates. Furthermore, since the Lie algebra of $\mathcal{G}(P)$ is the space of the equivariant, horizontal 0-forms over P, i.e $\Omega^0(\Sigma,\mathfrak{g}_P)$, in order to compute the infinitesimal gauge transformation on a connection A, we choose an element ϕ of the Lie algebra $\Omega^0(\Sigma,\mathfrak{g}_P)$ and we set $u_t = \exp(t\phi) = 1 + t\phi + O(t^2)$, then

$$\frac{d}{dt}\Big|_{t=0}(u_t^*A) = -\frac{d}{dt}\Big|_{t=0}(u_t^{-1}Au_t + u_t^{-1}du_t) = -[A,\phi] - d\phi = -d_A\phi. \tag{6}$$

In fact, choosing a connection $A \in \mathcal{A}(P)$, we can define the covariant derivative

$$d_A : \Omega^0(\Sigma,\mathfrak{g}_P) \to \Omega^1(\Sigma,\mathfrak{g}_P); \quad \phi \mapsto d_A\phi = d\phi + [A,\phi]$$

and the exterior derivative

$$d_A : \Omega^k(\Sigma,\mathfrak{g}_P) \to \Omega^{k+1}(\Sigma,\mathfrak{g}_P); \quad \omega \mapsto d_A\omega = d\omega + [A \wedge \omega]$$

where $[\omega_1 \wedge \omega_2] := \omega_1 \wedge \omega_2 - (-1)^{lk}\omega_2 \wedge \omega_1$ denotes the super Lie bracket operator for $\omega_1 \in \Omega^l(\Sigma,\mathfrak{g}_P)$ and $\omega_2 \in \Omega^k(\Sigma,\mathfrak{g}_P)$. The Hodge operator acts not only on $\Omega^k(\Sigma)$, but on $\Omega^k(\Sigma,\mathfrak{g}_P)$, too; in fact, since[1] $\Omega^k(\Sigma,\mathfrak{g}_P) = \Gamma\big(\wedge^k T^*\Sigma \otimes \mathfrak{g}_P\big)$, for all $\omega \in \Omega^k(\Sigma)$, and all $\xi \in \Omega^0(\Sigma,\mathfrak{g}_P)$, we define $*(\omega \otimes \xi) := *\omega \otimes \xi$. Therefore, using two inner products, one on $\Omega^k(\Sigma)$ defined using the Hodge operator and an invariant inner product on the Lie algebra on $\Omega^0(\Sigma,\mathfrak{g}_P)$, we have an inner product on the k-forms $\Omega^k(\Sigma,\mathfrak{g}_P)$

$$\langle a,b\rangle = \int_\Sigma \langle a \wedge *b\rangle \quad \forall a,b \in \Omega^k(\Sigma,\mathfrak{g}_P); \tag{7}$$

for two vector fields X, Y on Σ, $\langle a \wedge b\rangle(X,Y) = \langle a(X), b(Y)\rangle - \langle a(Y), b(X)\rangle$. Using this inner product we can define the adjoint operator

$$d_A^* : \Omega^{k+1}(\Sigma,\mathfrak{g}_P) \to \Omega^k(\Sigma,\mathfrak{g}_P)$$

[1] $\Gamma\big(\wedge^k T^*\Sigma \otimes \mathfrak{g}_P\big)$ denotes the sections of the bundle $\wedge^k T^*\Sigma \otimes \mathfrak{g}_P \to \Sigma$.

of the exterior derivative d_A, $A \in \mathcal{A}(P)$. For any connection $A \in \mathcal{A}(P)$, the two form $F_A := dA + \frac{1}{2}[A \wedge A] \in \Omega^2(\Sigma, \mathfrak{g}_P)$ is called curvature of A and the gauge group acts by $F_{u^*A} = u^{-1} F_A u$ for every $u \in \mathcal{G}(P)$. With this last definition it is possible to introduce the set of flat connections

$$\mathcal{A}_0(P) := \{A \in \mathcal{A}(P) \mid F_A = 0\}$$

and for an $A \in \mathcal{A}_0(P)$, since $d_A \circ d_A = 0$, the cohomology groups

$$H^k_A(\Sigma, \mathfrak{g}_P) := \ker d_A / \mathrm{im}\, d_A \Big|_{\Omega^k(\Sigma, \mathfrak{g}_P)} = \ker d_A \cap \ker d^*_A \Big|_{\Omega^k(\Sigma, \mathfrak{g}_P)}$$

are well defined for any $k \in \mathbb{N}$. Moreover, we have the orthogonal splitting

$$\Omega^k(\Sigma, \mathfrak{g}_P) = d_A \Omega^{k-1}(\Sigma, \mathfrak{g}_P) \oplus H^k_A(\Sigma, \mathfrak{g}_P) \oplus d^*_A \Omega^{k+1}(\Sigma, \mathfrak{g}_P) \tag{8}$$

and we denote the canonical projection in to the harmonic forms by π_A, i.e.

$$\pi_A : \Omega^k(\Sigma, \mathfrak{g}_P) \to H^k_A(\Sigma, \mathfrak{g}_P).$$

3 The moduli space $\mathcal{M}^g(P)$

For the following, we choose a compact oriented Riemann surface Σ of genus $g \geq 1$ and a non trivial principal SO(3)-bundle $\pi : P \to \Sigma$; next, we define the even gauge group $\mathcal{G}_0(P)$ as the unit component of $\mathcal{G}(P)$ and for more details we refer to [2]. Finally we can introduce the moduli space

$$\mathcal{M}^g(P) := \mathcal{A}_0(P)/\mathcal{G}_0(P)$$

which is a compact smooth manifold of dimension $6g - 6$ and if $g \geq 2$, then it is also connected and simply connected; these results were proved by Dostoglou and Salamon (cf. [2]) using the works of Newstead (cf. [11]).

Remark 3 If $g = 2$, then the moduli space $\mathcal{M}^2(P)$ can be seen as an intersection of quadrics in P_5 (cf. [12]).

Remark 4 Since an element $u \in \mathcal{G}_0(P)$, which is an element of the isotropy group[2], maps P to the identity, the operator $d_A : \Omega^0(\Sigma, \mathfrak{g}_P) \to \Omega^1(\Sigma, \mathfrak{g}_P)$ is injective. Moreover, $d^*_A d_A : \Omega^0(\Sigma, \mathfrak{g}_P) \to \Omega^0(\Sigma, \mathfrak{g}_P)$ is invertible, because the fact that d_A is injective implies that d^*_A is surjective by the decomposition (8) of $\Omega^0(\Sigma, \mathfrak{g}_p)$ and in addition $\mathrm{im}\, d^*_A = \mathrm{im}\, d^*_A d_A$ by the decomposition of $\Omega^1(\Sigma, \mathfrak{g}_p)$ (cf. [2] for more details).

The infinitesimal gauge transformation for $\Psi \in \Omega^0(\Sigma, \mathfrak{g}_P)$ acts on a connection by

$$\mathcal{A}(P) \to T\mathcal{A}(P); \quad A \mapsto -d_A \Psi$$

and thus, the tangent space at $[A] \in \mathcal{M}^g(P)$, $A \in \mathcal{A}_0(P)$, can be identified with the first homological group $H^1_A(\Sigma, \mathfrak{g}_P)$, in fact by (6) and by the orthogonal splitting $\ker d_A = \mathrm{im}\, d_A \oplus H^1_A(\Sigma, \mathfrak{g}_P)$, we have

$$T_A \mathcal{A}_0(P)/\mathrm{im}\, d_A = \ker d_A / \mathrm{im}\, d_A = H^1_A(\Sigma, \mathfrak{g}_P) \tag{9}$$

because the tangent space $T_A \mathcal{A}_0(P)$ corresponds to the kernel of d_A. Hence if we choose a conformal structure on Σ, then we have a complex structure on $\mathcal{M}^g(P)$ which is not, but the

[2] An $u \in \mathcal{G}(P)$ is an element of the isotropy group of a connection A if and only if $u^* A = A$.

Hodge-*- operator acting on $H^1_A(\Sigma, \mathfrak{g}_P)$. We refer to [2] and [9] for more details. Moreover, since the tangent space of $[A] \in \mathcal{M}^g(P)$, for every $A \in \mathcal{A}_0(P)$, can be identified with $H^1_A(\Sigma, \mathfrak{g}_P)$, we have a symplectic form $\omega_A(a, b) = \int_\Sigma \langle a \wedge b \rangle$, for $a, b \in H^1_A(\Sigma, \mathfrak{g}_P)$, and a complex structure defined by the Hodge-$*$-operator. Since the symplectic 2-form does not depend on the base connection A, it is constant and thus, closed. Hence, $\mathcal{M}^g(P)$ is a Kähler manifold; this symplectic approach of the space of connections was introduced by Atiyah and Bott in [1]. We conclude this section with the following result (cf. [8]).

Lemma 1 *We choose two flat connections $A', A'' \in \mathcal{A}_0(P)$, then*

$$\min_{u \in \mathcal{G}(P)} \|A' - u^* A''\|_{L^2(\Sigma)} \leq d([A'], [A''])$$

where $d(\cdot, \cdot)$ denotes the distance between $[A']$ and $[A'']$ on the smooth compact manifold $\mathcal{M}^g(P)$.

4 Perturbed geodesics on $\mathcal{M}^g(P)$

The idea is to find a loop $A \subset C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{A}_0)$ such that the projection $\Pi(A)$ on $\mathcal{M}^g(P)$ is a geodesic, where $\Pi : \mathcal{A}_0(P) \to \mathcal{M}^g(P)$, and since $\partial_t A \in T_A\mathcal{A}_0 = H^1_A(\Sigma, \mathfrak{g}_P) \oplus \operatorname{im} d_A$ and $d\Pi(A)\partial_t A \in T_{\Pi(A)}\mathcal{M}^g(P)$ which corresponds to $H^1_A(\Sigma, \mathfrak{g}_P)$,

$$0 = d_A^*(\partial_t A - d_A \Psi) = d_A(\partial_t A - d_A \Psi)$$

for a Ψ such that $\Psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ for all $t \in S^1$. Hence, since $d_A^* d_A$ is invertible, Ψ is uniquely determined and

$$\pi_A(\partial_t A) = \partial_t A - d_A(d_A^* d_A)^{-1} d_A^* \partial_t A = \partial_t A - d_A \Psi.$$

The unperturbed energy of our curve is, therefore,

$$E(A) = \frac{1}{2} \int_0^1 |d\Pi(A)\partial_t A|^2 dt = \frac{1}{2} \int_0^1 |\partial_t A - d_A \Psi|^2 dt. \tag{10}$$

If we consider a time dependent Hamiltonian map

$$\bar{H} : \mathbb{R}/\mathbb{Z} \times \mathcal{A}_0(P) \to \mathbb{R};\ (t, A) \mapsto \bar{H}_t(A)$$

which is invariant under the gauge group $\mathcal{G}_0(P)$ and constructed using the holonomy (cf. [2]); then we can perturbe the energy functional subtracting from E the integral of $\bar{H}_t(\cdot)$, i.e.

$$E^{\bar{H}}(A) = \frac{1}{2} \int_0^1 |\partial_t A - d_A \Psi|^2 dt - \int_0^1 \bar{H}_t(A) dt. \tag{11}$$

The equivariance of $\bar{H}_t(\cdot)$ means that we indroduce a perturbation on the energy functional on the loop space of the smooth manifold $\mathcal{M}^g(P)$. Weber [16] using the Thom–Smale transversality proved that the set

$$\nu_{reg} := \{H \in C^\infty(S^1 \times \mathcal{M}^g(P), \mathbb{R}) \mid \text{The Jacobi operator for } E^{\bar{H}} \text{ is bijective for all critical loops}\}$$

is open and dense in $C^\infty(S^1 \times \mathcal{M}^g(P), \mathbb{R})$ endowed with the compact-open topology and ν_{reg} is residual. Therefore we can choose $\bar{H}_t$ such that the Jacobi operator of $E^{\bar{H}}$ is invertible for all the perturbed geodesics. From now, we assume that our perturbation is chosen with this property. Furthermore, in the same paper Weber proved that below a given energy level we have only a finite number of perturbed geodesics.

Next, we extend the perturbation to the whole space of connections: $H_t : \mathcal{A}(P) \to \mathbb{R}$, where $H_t(A) = \bar{H}_t(A)$ for every $A \in \mathcal{A}_0(P)$. A first approach is to pick a gauge invariant holonomy perturbation on $\mathcal{A}(P)$ since every Hamiltonian H_t can be constructed in this way (cf. [2]); since H_t is constant along $\mathcal{G}(P)^*A$ for a given connection $A \in \mathcal{A}(P)$ and $T_A(\mathcal{G}(P)^*A) = \mathrm{im}\, d_A$,

$$d_A X_t(A) = 0. \tag{12}$$

Another possibility is the following. We pick a smooth map $\rho : [0, \infty) \to [0, 1]$ with the property that $\rho(x) = 0$ if $x \geq \delta_0^2$ and $\rho(x) = 1$ if $x \leq (\frac{2\delta_0}{3})^2$ for a δ_0 which satisfies the conditions of the Lemmas 11 and 12 for $p = 2$ and $q = 4$. Then we define $H_t(A) = 0$ for every A with $\|F_A\|_{L^2} \geq \delta_0$ and

$$H_t(A) := \rho\left(\|F_A\|_{L^2}^2\right) \bar{H}_t\left(A + *d_A\eta(A)\right)$$

otherwise, where $\eta(A)$ is the unique 0-form given by the Lemma 12 for the connection A. In this case, if A is flat then $H_t(A + *sd_A\eta)$ is constant for every 0-form $\eta \in \Omega^0(\Sigma, \mathfrak{g}_P)$ and every $s \in (-\varepsilon, \varepsilon)$ with ε sufficiently small and we can conclude that $d_A * X_t(A) = 0$. In both cases, the time-dependent Hamiltonian vector field $X_t : \mathcal{A}(P) \to \Omega^1(\mathfrak{g}_P)$ is defined such that, for any 1-form α and any connection A, $dH_t(A)\alpha = \int_\Sigma \langle X_t(A) \wedge \alpha\rangle$.

Theorem 3 *A closed curve A, $A(t) \in \mathcal{A}_0$ for all $t \in S^1 \cong \mathbb{R}/\mathbb{Z}$, descends to a perturbed geodesic if and only if there are $\Psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ and $\omega(t) \in \Omega^2(\Sigma, \mathfrak{g}_P)$ such that*

$$-\nabla_t(\partial_t A - d_A\Psi) - *X_t(A) - d_A^*\omega = 0, \tag{13}$$

$$d_A^*(\partial_t A - d_A\Psi) = 0, \tag{14}$$

where $\nabla_t := \partial_t + [\Psi, .]$. If this holds, ω is the unique solution of

$$d_A d_A^*\omega = [(\partial_t A - d_A\Psi) \wedge (\partial_t A - d_A\Psi)] - d_A * X_t(A). \tag{15}$$

Proof See [5]. □

Remark 5 We defined the moduli space of flat connections $\mathcal{M}^g(P)$ by taking the quotient $\mathcal{A}_0(P)/\mathcal{G}_0(P)$ where $\mathcal{G}_0(P)$ is the even gauge group and thus a geodesic $\gamma(t) \in \mathcal{M}^g(P)$ lifts to a closed path in $\mathcal{A}_0(P)$ which is unique modulo

$$\mathcal{G}_0\left(P \times S^1\right) := \{g \in \mathcal{G}(P \times S^1) \mid g(t) \in \mathcal{G}_0(P) \quad \forall t \in S^1\}.$$

The group $\mathcal{G}_0\left(P \times S^1\right)$ acts clearly also on the connections $\mathcal{A}(P \times S^1)$ of a principal bundle $P \times S^1 \to \Sigma \times S^1$.

We can therefore characterize the perturbed geodesics using the map

$$\mathcal{F}^0(A, \Psi) := \begin{pmatrix} -\nabla_t(\partial_t A - d_A\Psi) - *X_t(A) \\ -d_A^*(\partial_t A - d_A\Psi) \wedge dt \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1^0(A, \Psi) \\ \mathcal{F}_2^0(A, \Psi) \end{pmatrix} \tag{16}$$

defined for two loops $A(t) \in \mathcal{A}(P)$ and $\Psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$. In fact, a closed curve A, $A(t) \in \mathcal{A}_0(P)$ for all $t \in S^1 \cong \mathbb{R}/\mathbb{Z}$, descends to a perturbed geodesic if and only if $\mathcal{F}^0(A, \Psi) \in \operatorname{im} d_A^* \times \{0\}$. Next, we denote the set of perturbed geodesics below a energy level b by

$$\mathrm{Crit}^b_{E^H} := \left\{A + \Psi dt \in \mathcal{L}(\mathcal{A}_0(P) \otimes \Omega^0(\Sigma, \mathfrak{g}_P) \wedge dt) |\, E^H(A) \leq b, (13), (14)\right\}.$$

The Jacobi operator of a loop $A \subset \mathcal{A}_0$, which descends to a perturbed geodesic on $\mathcal{M}^g(P)$, is given by (cf. [5])

$$\begin{aligned}\mathcal{D}^0(A)(\alpha, \psi) = \pi_A &(2[\psi, (\partial_t A - d_A\Psi)] + d * X_t(A)\alpha + \nabla_t\nabla_t\alpha) \\ &+\pi_A \left(*\left[\alpha \wedge *d_A(d_A^* d_A)^{-1}\left(\nabla_t(\partial_t A - d_A\Psi) + *X_t(A)\right)\right]\right) \qquad (17)\end{aligned}$$

where $\alpha(t) \in H^1_{A(t)}(\Sigma, \mathfrak{g}_P)$, Ψ is defined uniquely by

$$d_A^*(\partial_t A - d_A\Psi) = 0 \tag{18}$$

and $\psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ by

$$-2 * [\alpha \wedge *(\partial_t A - d_A\Psi)] - d_A^* d_A\psi = 0. \tag{19}$$

5 Perturbed Yang–Mills connections

Now, we choose a Riemann metric g_Σ on the surface Σ and we consider the manifold $\Sigma \times S^1$ with the partial rescaled metric $(\varepsilon^2 g_\Sigma \oplus g_{S^1})$ for a given $\varepsilon \in]0, 1]$; furthermore, we denote by $\pi^\varepsilon : P \times S^1 \to \Sigma \times S^1$ the principal SO(3)-bundle over $\Sigma \times S^1$ and we assume that the restriction $P \times \{s\} \to \Sigma \times \{s\}$ is non trivial. If we choose a connection $\Xi = A + \Psi\, dt \in \mathcal{A}(P \times S^1)$ where $A(t) \in \mathcal{A}(P)$, $\Psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ for all $t \in S^1$, then the L^2-norm induced by the metric $(\varepsilon^2 g_\Sigma \oplus g_{S^1})$ of the curvature $F_\Xi = F_A - (\partial_t A - d_A\Psi) \wedge dt$ is given by

$$\|F_\Xi\|^2_{L^2} = \int_0^1 \left(\frac{1}{\varepsilon^2}\|F_A\|^2_{L^2(\Sigma)} + \|\partial_t A - d_A\Psi\|^2_{L^2(\Sigma)}\right) dt;$$

if we add the same perturbation as in (11), we can define the perturbed Yang–Mills functional

$$\mathcal{YM}^{\varepsilon,H}(\Xi) := \frac{1}{2}\int_0^1 \left(\frac{1}{\varepsilon^2}\|F_A\|^2_{L^2(\Sigma)} + \|\partial_t A - d_A\Psi\|^2_{L^2(\Sigma)}\right) dt - \int_0^1 H_t(A)\, dt. \tag{20}$$

A perturbed Yang–Mills connection is a critical connection $\Xi^\varepsilon = A^\varepsilon + \Psi^\varepsilon dt \in \mathcal{A}(P \times S^1)$ of $\mathcal{YM}^{\varepsilon,H}$ and has to satisfy the equation $d^{*\varepsilon}_{\Xi^\varepsilon} F_{\Xi^\varepsilon} - *X_t(A^v arepsilon) = 0$ that is equivalent to the two conditions

$$\frac{1}{\varepsilon^2} d^*_{A^\varepsilon} F_{A^\varepsilon} - \nabla_t(\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon) - *X_t(A^\varepsilon) = 0, \tag{21}$$

$$-\frac{1}{\varepsilon^2} d^*_{A^\varepsilon}(\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon) = 0. \tag{22}$$

In the following, if we write a perturbed Yang–Mills connection as $\Xi^\varepsilon = A^\varepsilon + \Psi^\varepsilon dt$ with apex ε, then we mean that Ξ^ε is a critical point of the functional $\mathcal{YM}^{\varepsilon,H}$ and we denote the set of perturbed Yang–Mills connections below an energy level b by

$$\mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}} := \left\{\Xi^\varepsilon \in \mathcal{A}(P \times S^1) |\, \mathcal{YM}^{\varepsilon,H}\left(\Xi^\varepsilon\right) \leq b, (21), (22)\right\}.$$

If we fix a connection $\Xi^0 = A^0 + \Psi^0 dt$, then we can define an ε-dependent map $\mathcal{F}^\varepsilon$, for $\varepsilon > 0$, by $\mathcal{F}^\varepsilon(A, \Psi) = \mathcal{F}_1^\varepsilon(A, \Psi) + \mathcal{F}_2^\varepsilon(A, \Psi)$ where

$$\begin{aligned}\mathcal{F}_1^\varepsilon(A, \Psi) &= \frac{1}{\varepsilon^2} d_A^* F_A - \nabla_t(\partial_t A - d_A \Psi) - *X_t(A)\\ &\quad + \frac{1}{\varepsilon^2} d_A d_A^*(A - A^0) - d_A \nabla_t(\Psi - \Psi^0), \end{aligned} \tag{23}$$

$$\mathcal{F}_2^\varepsilon(A, \Psi) = \left(-\frac{1}{\varepsilon^2} d_A^*(\partial_t A - d_A \Psi) + \frac{1}{\varepsilon^2} \nabla_t d_A^*(A - A^0) - \nabla_t^2(\Psi - \Psi^0) \right) \wedge dt; \tag{24}$$

then the zeros of $\mathcal{F}^\varepsilon$ are perturbed ε-Yang–Mills connections and they satisfy the local gauge condition

$$d_{\Xi^0}^{*_\varepsilon}\left(\Xi - \Xi^0\right) = \frac{1}{\varepsilon^2} d_{A^0}^*(A - A^0) - \nabla_t^{\Psi^0}(\Psi - \Psi^0) = 0$$

respect to the reference connection $A^0 + \Psi^0 dt$ by the following remark already considered by Hong (cf. [5]).

Remark 6 $\Xi^\varepsilon = A^\varepsilon + \Psi^\varepsilon dt$ is a perturbed Yang–Mills connection on $P \times S^1$ and satisfies the gauge condition $d_{\Xi^\varepsilon}^{*_\varepsilon}(\alpha^\varepsilon + \psi^\varepsilon dt) = 0$ with $\alpha^\varepsilon + \psi^\varepsilon dt := \Xi^\varepsilon - \Xi^0$ if and only if

$$d_{\Xi^\varepsilon} d_{\Xi^\varepsilon}^{*_\varepsilon}(\alpha^\varepsilon + \psi^\varepsilon dt) + d_{\Xi^\varepsilon}^{*_\varepsilon} F_{\Xi^\varepsilon} - *X_t(A^\varepsilon) = 0. \tag{25}$$

One can see this deriving (25) by $d_{\Xi^\varepsilon}^{*_\varepsilon}$.

Remark 7 If we choose $\frac{3}{2} < p < \infty$ and $b > 0$, then for every perturbed Yang–Mills connection $\Xi^\varepsilon = A^\varepsilon + \Psi^\varepsilon dt \in \mathcal{A}^{1,p}(P \times S^1)$, there exists a gauge transformation $u \in \mathcal{G}_0^{2,p}(P \times S^1)$ such that $u^* \Xi^\varepsilon$ is smooth. A proof of this statement for weak Yang–Mills connections can be found in [17] (cf. theorem 9.4) and the proof holds also for perturbed Yang–Mills connections.

If we linearise the Eqs. (21) and (22) we obtain the two components of the Jacobi operator

$$\mathrm{Jac}^{\varepsilon,H}(\Xi^\varepsilon) : \Omega^1(\Sigma \times S^1, \mathfrak{g}_P) \to \Omega^1(\Sigma \times S^1, \mathfrak{g}_P)$$

of a perturbed Yang–Mills connection:

$$\begin{aligned}&\mathrm{Jac}^{\varepsilon,H}(A^\varepsilon + \Psi^\varepsilon dt)(\alpha, \psi)\\ &\quad = \frac{1}{\varepsilon^2} d_{A^\varepsilon}^* d_{A^\varepsilon}\alpha + \frac{1}{\varepsilon^2} * [\alpha \wedge *F_{A^\varepsilon}] - d * X_t(A^\varepsilon)\alpha\\ &\quad\quad - \nabla_t \nabla_t \alpha + d_{A^\varepsilon} \nabla_t \psi - 2[\psi, (\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon)]\\ &\quad\quad + \left(\frac{1}{\varepsilon^2} * [\alpha \wedge *(\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon)] - \frac{1}{\varepsilon^2} \nabla_t d_{A^\varepsilon}^* \alpha + \frac{1}{\varepsilon^2} d_{A^\varepsilon}^* d_{A^\varepsilon} \psi \right) dt, \end{aligned} \tag{26}$$

for any $\alpha(t) \in \Omega^1(\Sigma, \mathfrak{g}_P)$ and $\psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$.

In 1982, Atiyah and Bott (cf. [1]) showed that the Jacobi operator of a Yang–Mills connection $\Xi^\varepsilon = A^\varepsilon + \Psi^\varepsilon dt$ is Fredholm of index 0; for the perturbed case we have the same result. First, we recall that the gauge group acts on the 1-forms adding the image of d_{Ξ^ε} and hence $\alpha + \psi\, dt$ is an element of $\Omega^1(\Sigma \times S^1, \mathfrak{g}_P)/\mathcal{G}_\Sigma(P \times S^1)$ if and only if

$$0 = \langle \alpha + \psi\, dt, d_{\Xi^\varepsilon}\phi \rangle = \langle d_{\Xi^\varepsilon}^{*_\varepsilon}(\alpha + \psi\, dt), \phi \rangle$$

for every $\phi \in \Omega^0(\Sigma \times S^1, \mathfrak{g}_P)$ and consequently, if and only if $\alpha + \psi\, dt \in \ker d_{\Xi^\varepsilon}^{*_\varepsilon}$. Therefore, under the condition $d_{\Xi^\varepsilon}^{*_\varepsilon}(\alpha + \psi\, dt) = 0$ we have that

$$\mathrm{Jac}^{\varepsilon,H}(\Xi^\varepsilon)(\alpha + \psi\, dt) = \mathrm{Jac}^{\varepsilon,H}(\Xi^\varepsilon)(\alpha + \psi\, dt) + d_{\Xi^\varepsilon} d_{\Xi^\varepsilon}^*(\alpha + \psi\, dt) \tag{27}$$

which can be written as

$$\left(d^{*_\varepsilon}_{\Xi^\varepsilon} d_{\Xi^\varepsilon} + d_{\Xi^\varepsilon} d^{*_\varepsilon}_{\Xi^\varepsilon}\right)(\alpha + \psi\, dt) + *[(\alpha + \psi\, dt) \wedge *F_{\Xi^\varepsilon}] - d * X_t(A^\varepsilon)(\alpha + \psi\, dt) \tag{28}$$

where the first term is the Laplace operator of $\alpha + \psi\, dt$ and the second one is of order zero and thus, we have a selfadjoint elliptic operator and therefore, a Fredholm operator with index 0. In addition, this can allow us to work with (28) instead of using the Jacobi operator and (28) can be written as the operator $\mathcal{D}^\varepsilon(A + \Psi dt) := \mathcal{D}_1^\varepsilon(A + \Psi dt) + \mathcal{D}_2^\varepsilon(A + \Psi dt)\, dt$ given by

$$\begin{aligned}\mathcal{D}_1^\varepsilon(A + \Psi dt)(\alpha, \psi) :=& \frac{1}{\varepsilon^2}\left(d_A^* d_A \alpha + d_A d_A^* \alpha + *[\alpha \wedge *F_A]\right) - d * X_t(A)\alpha \\ & - \nabla_t \nabla_t \alpha - 2[\psi, (\partial_t A - d_A \Psi)] \\ \mathcal{D}_2^\varepsilon(A + \Psi dt)(\alpha, \psi) :=& \frac{1}{\varepsilon^2}\left(2 * [\alpha \wedge *(\partial_t A - d_A \Psi)] + d_A^* d_A \psi\right) - \nabla_t \nabla_t \psi.\end{aligned} \tag{29}$$

Moreover, the operator $\mathcal{D}^\varepsilon$ is almost the linearisation of $\mathcal{F}^\varepsilon$; to be precise $\mathcal{D}^\varepsilon$ does not contain the derivatives of d_A, d_A^* and ∇_t of the last two terms in both components (23) and (24), because these can be treated like quadratic terms as we will see in the Lemma 5. If the reference connection $A + \Psi\, dt$ is clear from the context, then we will write the operators without indicating it.

6 Norms

If we fix a connection $\Xi_0 = A_0 + \Psi_0 dt \in \mathcal{A}(\Sigma \times S^1)$, then we can define a norm on its tangential space and since $\mathcal{A}(\Sigma \times S^1)$ is an affine space, we can use it as a metric on $\mathcal{A}(\Sigma \times S^1)$. Let be $\xi(t) = \alpha(t) + \psi(t) \wedge dt$ such that $\alpha(t) \in \Omega^1(\Sigma, \mathfrak{g}_P)$ and $\psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ or $\alpha(t) \in \Omega^2(\Sigma, \mathfrak{g}_P)$ and $\psi(t) \in \Omega^1(\Sigma, \mathfrak{g}_P)$. Then we define the following norms

$$\begin{aligned}\|\xi\|^p_{0,p,\varepsilon} &:= \int_0^1 \left(\|\alpha\|^p_{L^p(\Sigma)} + \varepsilon^p \|\psi\|^p_{L^p(\Sigma)}\right) dt, \\ \|\xi\|_{\infty,\varepsilon} &:= \|\alpha\|_{L^\infty(\Sigma\times S^1)} + \varepsilon\|\psi\|_{L^\infty(\Sigma\times S^1)}\end{aligned}$$

and

$$\begin{aligned}\|\xi\|^p_{\Xi_0,1,p,\varepsilon} :=& \int_0^1 \left(\|\alpha\|^p_{L^p(\Sigma)} + \|d_{A_0}\alpha\|^p_{L^p(\Sigma)} + \|d^*_{A_0}\alpha\|^p_{L^p(\Sigma)} + \varepsilon^p \|\nabla_t \alpha\|^p_{L^p(\Sigma)}\right) dt \\ & + \int_0^1 \varepsilon^p \left(\|\psi\|^p_{L^p(\Sigma)} + \|d_{A_0}\psi\|^p_{L^p(\Sigma)} + \varepsilon^p \|\nabla_t \psi\|^p_{L^p(\Sigma)}\right) dt, \\ \|\psi\|^p_{0,p,\varepsilon} :=& \int_0^1 \|\psi\|^p_{L^p(\Sigma)}\, dt.\end{aligned}$$

Inductively,

$$\begin{aligned}\|\xi\|^p_{\Xi_0,k+1,p,\varepsilon} := &\|\alpha+\psi\,dt\|^p_{\Xi_0,k,p,\varepsilon}+\|d_{A_0}\alpha\|^p_{\Xi_0,k,p,\varepsilon}+\|d^*_{A_0}\alpha\|^p_{\Xi_0,k,p,\varepsilon}\\ &+\varepsilon^p\|\nabla_t\alpha\|^p_{\Xi_0,k,p,\varepsilon}+\|d_{A_0}\psi\wedge dt\|^p_{\Xi_0,k,p,\varepsilon}+\varepsilon^p\|\nabla_t\psi\,dt\|^p_{\Xi_0,k,2,\varepsilon}.\end{aligned}$$

Also in this case, if the reference connection is clear from the context we write the norms without mentioning it.

Remark 8 For $i=1,2$, we define by $W^{k,p}(\Sigma\times S^1,\Lambda^iT^*(\Sigma\times S^1)\otimes\mathfrak{g}_{P\times S^1})$ the Sobolev space of the sections of $\Lambda^iT^*(\Sigma\times S^1)\otimes\mathfrak{g}_{P\times S^1}\to\Sigma\times S^1$ as the completion of[3]

$$\Gamma(\Lambda^iT^*(\Sigma\times S^1)\otimes\mathfrak{g}_P)=\Omega^i(\Sigma\times S^1,\mathfrak{g}_{P\times S^1})$$

respect to the norm $\|\cdot\|_{\Xi_0,k,p,1}$. Furthermore, we can define the Sobolev space of the connections on $P\times S^1$ as[4]

$$\mathcal{A}^{k,p}(P\times S^1)=\Xi_0+W^{k,p},$$

where $W^{k,p}=W^{k,p}(\Sigma\times S^1,T^*(\Sigma\times S^1)\otimes\mathfrak{g}_{P\times S^1})$, $\Xi_0\in\mathcal{A}(P\times S^1)$.

Remark 9 The Sobolev space of gauge transformations $\mathcal{G}_0^{2,p}(P\times S^1)$ is the completion of $\mathcal{G}_0(P\times S^1)$ with respect the Sobolev $W^{1,p}$-norm on 1-forms, i.e. $g\in\mathcal{G}_0^{2,p}(P\times S^1)$ if $g^{-1}d_{\Sigma\times S^1}g\in W^{1,p}$ and hence $g:\mathcal{A}^{1,p}(P\times S^1)\to\mathcal{A}^{1,p}(P\times S^1)$.

Remark 10 The gauge condition $d^*_{\Xi^\varepsilon}(\Xi^\varepsilon-\Xi_0)=0$ assures us that if the perturbed Yang–Mills connection Ξ^ε is an element of $\mathcal{A}^{1,2}(P\times S^1)$, then, for any $k\geq 2$, there is an $u\in\mathcal{G}_0^{2,p}(P\times S^1)$ such that $u^*\Xi^\varepsilon\in\mathcal{A}^{k,2}(P)$ (cf. [17], Chapter 9).

We now choose a reference connection Ξ_0 and analogously as for the lemma 4.1 in [3], if we define $\bar\xi=\bar\alpha+\bar\psi\,dt$ where $\bar\alpha(t)=\alpha(\varepsilon s)$ and $\bar\psi(t)=\varepsilon\psi(\varepsilon s), 0\leq t\leq\varepsilon^{-1}$, then $\|\xi\|_{k,p,\varepsilon}=\varepsilon^{\frac{1}{p}}\|\bar\xi\|_{W^{k,p}}$. In addition, all the Sobolev inequalities hold as follows by the Sobolev embedding theorem (cf. theorem B2 in [17]).

Theorem 4 (Sobolev estimates) *We choose $1\leq p,q<\infty$ and $l\leq k$. Then there is a constant c_s such that for every $\xi\in W^{k,p}(\Sigma\times S^1,\Lambda^iT^*(\Sigma\times S^1)\otimes\mathfrak{g}_{P\times S^1})$, $i=1,2$, and any reference connection Ξ_0:*

1. *If $l-\frac{3}{q}\leq k-\frac{3}{p}$, then*

$$\|\xi\|_{\Xi_0,l,q,\varepsilon}\leq c_s\varepsilon^{1/q-1/p}\,\|\xi\|_{\Xi_0,k,p,\varepsilon}. \tag{30}$$

2. *If $0<k-\frac{3}{p}$, then*

$$\|\xi\|_{\Xi_0,\infty,\varepsilon}\leq c_s\varepsilon^{-1/p}\,\|\xi\|_{\Xi_0,k,p,\varepsilon}. \tag{31}$$

[3] Let $E\to M$ a vector bundle, then ΓE denotes the space of section of the bundle.

[4] For more information, see appendix B of [17].

7 Elliptic estimates

The aim of this chapter is to estimate (Theorem 6) the $\|\cdot\|_{2,p,\varepsilon}$-norm of a 1-form $\xi = \alpha + \psi\, dt$ using the L^p-norm of the operator $\mathcal{D}^\varepsilon(\Xi)$ when $\Xi = A + \Psi dt$ represents a perturbed closed geodesic on $\mathcal{M}^g(P)$. We recall that we assume the Jacobi operator to be invertible for every perturbed geodesic. Hong [5] proved a weaker estimate which, in our setting, can be identified with

$$\begin{aligned}&\|\alpha + \psi\, dt - \pi_A(\alpha)\|_{1,2,\varepsilon} + \varepsilon\|\pi_A(\alpha)\|_{1,2,\varepsilon}\\ &\quad\le c\varepsilon^2\|\mathcal{D}^\varepsilon(\Xi + \alpha_0^\varepsilon)(\alpha,\psi)\|_{0,2,\varepsilon} + c\varepsilon\|\pi_A\mathcal{D}^\varepsilon(\Xi + \alpha_0^\varepsilon)(\alpha,\psi)\|_{L^2}\end{aligned}$$

where $\alpha_0^\varepsilon \in \operatorname{im} d_A^*$ is the unique solution of

$$d_A^* d_A \alpha_0^\varepsilon = \varepsilon^2 \nabla_t(\partial_t A - d_A\Psi) + *X_t(A^0);$$

in addition, he estended the last estimate to

$$\|\alpha + \psi\, dt\|_{k,2,\varepsilon} \le c\|\mathcal{D}^\varepsilon(\Xi + \alpha_0^\varepsilon)(\alpha,\psi)\|_{k-1,2,\varepsilon}$$

and with this inequality he proved the existence of a map from the set of the perturbed geodesics $\mathrm{Crit}^b_{E^H}$ to the set of the perturbed Yang–Mills connections $\mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$, but he did not show its uniqueness and its surjectivity. With the last two estimates is not possible to obtain the uniqueness statement of the Theorem 9 even for $p = 2$ and, as we have already discussed, the surjectivity could not be established using his rescaling of the metric, in particular because you can not expect that the norms of the curvature $\partial_t A - d_A\Psi$ have a uniforme bound for all the Yang–Mills connections below a given energy level.

For this chapter we choose a regular value b of the energy E^H, we fix a perturbed closed geodesic $\Xi = A + \Psi\, dt \in \mathrm{Crit}^b_{E^H}$ and we define every operator and every norm using this connection. Since the perturbed geodesic Ξ is smooth, there is a positive constant c_0 which bounds the L^∞-norm of the velocity and its derivatives, in particular

$$\|\partial_t A - d_A\Psi\|_{L^\infty} + \|\nabla_t(\partial_t A - d_A\Psi)\|_{L^\infty} \le c_0. \tag{32}$$

In general, we denote a constant, which is needed to fulfill an estimate, by c; it can therefore indicate different constants also in a single computation.

Theorem 5 *We choose a constant $p \ge 2$. If $p = 2$ we set $j = 0$ otherwise $j = 1$. There exist two constants $\varepsilon_0 > 0$ and $c > 0$ such that*

$$\|\xi\|_{2,p,\varepsilon} \le c\left(\varepsilon\|\mathcal{D}^\varepsilon(\xi)\|_{0,p,\varepsilon} + \|\pi_A(\alpha)\|_{L^p}\right), \tag{33}$$

$$\|(1-\pi_A)\xi\|_{2,p,\varepsilon} \le c\left(\varepsilon^2\|\mathcal{D}^\varepsilon(\xi)\|_{0,p,\varepsilon} + \varepsilon\|\pi_A(\alpha)\|_{L^p} + j\varepsilon^2\|\nabla_t^2\pi_A(\alpha)\|_{L^p}\right), \tag{34}$$

$$\begin{aligned}\|\alpha - \pi_A(\alpha)\|_{2,p,\varepsilon} &\le c\varepsilon^2\left(\|\mathcal{D}_1^\varepsilon(\xi)\|_{L^p} + \varepsilon^2\|\mathcal{D}_2^\varepsilon(\xi)\|_{L^p}\right)\\ &\quad + c\varepsilon^2\left(\|\pi_A(\alpha)\|_{L^p} + \|\nabla_t^{j+1}\pi_A(\alpha)\|_{L^p}\right),\end{aligned} \tag{35}$$

for every $\xi = \alpha + \psi\, dt \in \mathrm{W}^{2,p}$ and $0 < \varepsilon < \varepsilon_0$.

We want also to remark that the estimates for $p = 2$ are enough to prove the bijection between the critical connections, but for the identification between the flows between the critical points, which is discussed in [7], we need the theorem also for $p > 2$. We recall that by the Lemma 3 for perturbed geodesic $\Xi = A + \Psi dt$ we can associate a two form ω defined as the unique solution of

$$d_A d_A^*\omega = [(\partial_t A - d_A\Psi) \wedge (\partial_t A - d_A\Psi)]$$

which is equivalent to

$$\omega = d_A(d_A^* d_A)^{-1}(\nabla_t(\partial_t A - d_A\Psi) + *X_t(A)).$$

Theorem 6 *We choose $p \geq 2$ and we assume that there is a constant c_0 such that*

$$|\langle \mathcal{D}^0(\bar{\alpha}), \bar{\alpha}\rangle| \geq c_0 \left(\|\bar{\alpha}\|_{L^2} + \|\nabla_t\bar{\alpha}\|_{L^2}\right)^2 \tag{36}$$

for every $\bar{\alpha} \in W^{2,p}$. Then there are two constants $c > 0$ and $\varepsilon_0 > 0$ such that

$$\begin{aligned}&\|\pi_A(\alpha)\|_{L^p} + \|\nabla_t\pi_A(\alpha)\|_{L^p} + \|\nabla_t^2\pi_A(\alpha)\|_{L^p}\\&\quad \leq c\big(\varepsilon\|\mathcal{D}^\varepsilon(\alpha,\psi)\,dt\|_{0,p,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^p}\big),\end{aligned} \tag{37}$$

$$\|\alpha + \psi\,dt\|_{2,p,\varepsilon} \leq c\big(\varepsilon\,\|\mathcal{D}^\varepsilon(\alpha,\psi)\|_{0,p,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^p}\big), \tag{38}$$

$$\begin{aligned}&\|\alpha + \psi\,dt - \pi_A(\alpha)\|_{2,p,\varepsilon}\\&\quad \leq c\left(\varepsilon^2\|\mathcal{D}^\varepsilon(\alpha,\psi)\|_{0,p,\varepsilon} + \varepsilon\|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^p}\right),\end{aligned} \tag{39}$$

$$\begin{aligned}\|\alpha - \pi_A(\alpha)\|_{2,p,\varepsilon} &\leq c\varepsilon^2\|\mathcal{D}_1^\varepsilon(\alpha,\psi)\|_{L^p} + c\varepsilon^4\|\mathcal{D}_2^\varepsilon(\alpha,\psi)\|_{L^p}\\&\quad + c\varepsilon^2\|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^p}\end{aligned} \tag{40}$$

for every $\alpha + \psi\,dt \in W^{2,p}$ and $0 < \varepsilon < \varepsilon_0$.

Remark 11 The condition (36) is always satisfied whenever the Jacobi operator $\mathcal{D}^0$ is invertible because there is a positive constant c sucht that $\|\bar{\alpha}\|_{L^2}^2 \leq c\left\langle \mathcal{D}^0(\bar{\alpha}), \bar{\alpha}\right\rangle_{L^2}$ and

$$\|\nabla_t\bar{\alpha}\|_{L^2}^2 = \left|\langle \pi_A(\nabla_t\nabla_t\bar{\alpha}), \bar{\alpha}\rangle_{L^2}\right| \leq c\left|\left\langle \mathcal{D}^0(\bar{\alpha}), \bar{\alpha}\right\rangle_{L^2}\right| + c\|\bar{\alpha}\|_{L^2}^2$$

where the last estimate follows from the definition of $\mathcal{D}^0(\bar{\alpha})$ and (32).

We first prove the Theorem 6 using the Theorem 5 which will be discussed later.

Proof (*Theorem* 6) In order to prove the theorem we start with the estimates proved by Hong (cf. [5])

$$\begin{aligned}&\|\pi_A(\alpha)\|_{L^2} + \|\nabla_t\pi_A(\alpha)\|_{L^2}\\&\quad \leq c\|\pi_A\big(\mathcal{D}_1^{\varepsilon_\nu}(\Xi^0)(\alpha,\psi) + *[\alpha\wedge *\omega]\big)\|_{L^2} + c\|(1-\pi_A)(\alpha)\|_{L^2}\\&\qquad + c\|\nabla_t(1-\pi_A)(\alpha)\|_{L^2} + c\varepsilon^2\|\nabla_t\psi\|_{L^2} + \varepsilon^2\|\psi\|_{L^2}\\&\qquad + c\varepsilon_\nu^2\|\mathcal{D}_2^{\varepsilon_\nu}(\Xi^0)(\alpha,\psi)\|_{L^2},\end{aligned} \tag{41}$$

$$\begin{aligned}&\|\pi_A(\alpha)\|_{L^2} + \|\nabla_t\pi_A(\alpha)\|_{L^2}\\&\quad \leq c\left(\varepsilon\|\mathcal{D}^\varepsilon(\alpha,\psi)\|_{0,2,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^2}\right)\end{aligned} \tag{42}$$

and in addition for $q \geq 2$, using the orthogonal splitting and the invertibility property operators $d_A d_A^*$ and $d_A^* d_A$, we have that

$$\begin{aligned}\|\nabla_t\nabla_t\pi_A(\alpha)\|_{L^q} &\leq c\left\|(d_A + d_A^*)\nabla_t\nabla_t\pi_A(\alpha)\right\|_{L^q} + \|\pi_A\nabla_t\nabla_t\pi_A(\alpha)\|_{L^q}\\&\leq c\,\|\nabla_t\pi_A(\alpha)\|_{L^q} + c\,\|\pi_A(\alpha)\|_{L^q}\\&\quad + \left\|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha + \psi dt) - *[\alpha, *\omega]\right)\right\|_{L^q}\\&\quad + c\|\alpha\|_{L^q} + c\|\psi\|_{L^q} + c\|\nabla_t(1-\pi_A)\alpha\|_{L^q}\\&\leq c\left(\varepsilon\|\mathcal{D}^\varepsilon(\alpha,\psi)\|_{0,2,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^2}\right)\end{aligned} \tag{43}$$

where the second inequality follows from the commutation formulas, the definition of $\mathcal{D}_1^\varepsilon$ and the triangular inequality and the third by the Theorem 5 and (42). Finally, in the case $p = 2$, the Theorem 6 follows from the Theorem 5 and from the inequalities (42) and (43) for $q = 2$. For $2 < p < 6$ we use the Sobolev's Theorem 4 for $\varepsilon = 1$:

$$\begin{aligned}
&\|\pi_A(\alpha)\|_{L^p} + \|\nabla_t \pi_A(\alpha)\|_{L^p} \\
&\quad \le c\left(\|\pi_A(\alpha)\|_{L^2} + \|\nabla_t \pi_A(\alpha)\|_{L^2} + \|\nabla_t^2 \pi_A(\alpha)\|_{L^2}\right) \\
&\quad \le c\Big(\varepsilon\|\mathcal{D}^\varepsilon(\alpha,\psi)\,dt\|_{0,2,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^2}\Big) \\
&\quad \le c\Big(\varepsilon\|\mathcal{D}^\varepsilon(\alpha,\psi)\,dt\|_{0,p,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^p}\Big),
\end{aligned} \tag{44}$$

where the third step follows from the Hölder identity. (43), (44) and the Theorem 5 yield now to the estimates (38), (39) and (40). The estimate (37) follows then from (43) with $q = p$, (38) and (39). In order to prove the estimates for $p \ge 6$ we proceed in the same way. By the Sobolev's Theorem 4 for $\varepsilon = 1$ and the Hölder inequality:

$$\begin{aligned}
&\|\pi_A(\alpha)\|_{L^p} + \|\nabla_t \pi_A(\alpha)\|_{L^p} \\
&\quad \le c\left(\|\pi_A(\alpha)\|_{L^3} + \|\nabla_t \pi_A(\alpha)\|_{L^3} + \|\nabla_t^2 \pi_A(\alpha)\|_{L^3}\right) \\
&\quad \le c\Big(\varepsilon\|\mathcal{D}^\varepsilon(\alpha,\psi)\,dt\|_{0,3,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^3}\Big) \\
&\quad \le c\Big(\varepsilon\|\mathcal{D}^\varepsilon(\alpha,\psi)\,dt\|_{0,p,\varepsilon} + \|\pi_A\left(\mathcal{D}_1^\varepsilon(\alpha,\psi) + *[\alpha\wedge *\omega]\right)\|_{L^p}\Big).
\end{aligned} \tag{45}$$

The estimates (38), (39) and (40) are a consequense of (45) and the Theorem (5); (37) follows then from (43) with $q = p$, (38) and (39). This proves the theorem. □

Lemma 2 *We have the following two commutation formulas:*

$$[d_A, \nabla_t] = -[(\partial_t A - d_A\Psi)\wedge\cdot\,], \tag{46}$$

$$[d_A^*, \nabla_t] = *[(\partial_t A - d_A\Psi)\wedge *\cdot\,]. \tag{47}$$

Proof The lemma follows from the definitions of the operators using the Jacoby identity for the super Lie bracket operator. □

In the following pages we prepare the proof of the Theorem 5 and in order to do this we start showing the next result.

Theorem 7 *For $1 < p < \infty$ there exist two constants $\varepsilon_0 > 0$ and $c > 0$ such that*

$$\|\psi\|_{2,p,\varepsilon} \le c\left(\|(d_A^* d_A - \varepsilon^2\nabla_t\nabla_t)\psi\|_{L^p} + \|\psi\|_{1,p,\varepsilon}\right) \tag{48}$$

$$\|\alpha\|_{2,p,\varepsilon} \le c\left(\|(d_A d_A^* + d_A^* d_A - \varepsilon^2\nabla_t\nabla_t)\alpha\|_{L^p} + \|\alpha\|_{1,p,\varepsilon}\right) \tag{49}$$

for every 1-form $\alpha \in \mathrm{W}^{2,p}$ and every 0-form $\psi \in \mathrm{W}^{2,p}$, $0 < \varepsilon < \varepsilon_0$.

Proof We prove the theorem in four steps and in the first three we work in local coordinates and hence we consider the following setting. We choose a metric $g = g_{\mathbb{R}^2} \oplus dt^2$ on $U \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$ with U open and contained in a compact set, a constant connection $\Xi_c = A_c + \Psi_c dt \in \Omega^1(U \times \mathbb{R}, \mathfrak{g})$ of the trivial bundle $U \times \mathbb{R} \times \mathbf{SO}(3) \to U \times \mathbb{R}$ which satisfies $F_{A_c} = 0$ and a positive constant c_0. Furthermore we pick a connection $\tilde{\Xi} = \tilde{A} + \tilde{\Psi}dt \in \Omega^1(U \times \mathbb{R}, \mathfrak{g})$ which satisfies

$$\begin{aligned}
\|(\tilde{A} - A) + (\tilde{\Psi} - \Psi)\,dt\|_{\infty,\varepsilon} + \|d_A^*(\tilde{A} - A)\|_{L^\infty} &\le c_0, \\
\|d_A(\tilde{A} - A) + d_A(\tilde{\Psi} - \Psi)\,dt\|_{\infty,\varepsilon} &\le c_0, \\
\varepsilon\|\nabla_t(\tilde{A} - A) + \nabla_t(\tilde{\Psi} - \Psi)\,dt\|_{\infty,\varepsilon} &\le c_0.
\end{aligned} \tag{50}$$

Step 1. For $1 < p < \infty$ there exists a constant c, such that

$$\|\psi\|_{W^{2,p}} \leq c\left(\|d^* d\psi\|_{L^p} + \|\psi\|_{W^{1,p}}\right) \tag{51}$$
$$\|\alpha\|_{W^{2,p}} \leq c\left(\|(d^*d + dd^*)\alpha\|_{L^p} + \|\alpha\|_{W^{1,p}}\right) \tag{52}$$

hold for every 0-form with compact support $\psi \in W_c^{2,p}(U \times \mathbb{R}, \mathfrak{g})$ and every 1-form $\alpha \in W_c^{2,p}(U \times \mathbb{R}, T^*(U \times \mathbb{R}) \times \mathfrak{g})$ with compact support in $U \times \mathbb{R}$.

Proof The first step follows directly from the Calderon-Zygmund inequality, i.e.

$$\|u\|_{W^{2,p}} \leq c\left(\|\Delta_g u\|_{L^p} + \|u\|_{W^{1,p}}\right)$$

for every $u \in W_c^{2,p}(U \times \mathbb{R})$ with compact support in $U \times \mathbb{R}$. We refer to the chapter 2 and 3 of [17] for the details. □

Step 2. For $1 < p < \infty$ there exists a constant c, such that

$$\|\psi\|_{\Xi_c,2,p,\varepsilon} \leq c\left(\left\|d_{A_c}^* d_{A_c}\psi - \varepsilon^2 \nabla_t^{\Psi_c}\nabla_t^{\Psi_c}\psi\right\|_{L^p} + \|\psi\|_{\Xi_c,1,p,\varepsilon}\right) \tag{53}$$
$$\|\alpha\|_{\Xi_c,2,p,\varepsilon} \leq c\left(\left\|\left(d_{A_c}^* d_{A_c} + d_{A_c}d_{A_c}^* - \varepsilon^2 \nabla_t^{\Psi_c}\nabla_t^{\Psi_c}\right)\alpha\right\|_{L^p} + \|\alpha\|_{\Xi_c,1,p,\varepsilon}\right) \tag{54}$$

holds for every 0-form $\psi \in W_c^{2,p}(U \times \mathbb{R}, \mathfrak{g})$ and every 1-form $\alpha \in W_c^{2,p}(U \times \mathbb{R}, T^*(U \times \mathbb{R}) \times \mathfrak{g})$ with compact support in $U \times \mathbb{R}$.

Proof First, since the norms $\|\cdot\|_{W^{i,p}}$ and $\|\cdot\|_{\Xi_c,i,p,1}$ are equivalent

$$\begin{aligned}
\|\psi\|_{\Xi_c,2,p,1} &\leq \|\psi\|_{W^{2,p}} + c\|\Xi_c\|_{C^1}\|\psi\|_{W^{1,p}}\\
&\leq c\left(\|(d^*d)\psi\|_{L^p} + c\|\psi\|_{\Xi_c,1,p,1} + c\|\Xi_c\|_{L^\infty}\|\psi\|_{L^p}\right)\\
&\leq c\left(\left\|d_{A_c}^* d_{A_c}\psi - \nabla_t^{\Psi_c}\nabla_t^{\Psi_c}\psi\right\|_{L^p} + (1 + \|\Xi_c\|_{C^1})\|\psi\|_{\Xi_c,1,p,1}\right)\\
&\leq c\left(\left\|d_{A_c}^* d_{A_c}\psi - \nabla_t^{\Psi_c}\nabla_t^{\Psi_c}\psi\right\|_{L^p} + \|\psi\|_{\Xi_c,1,p,1}\right)
\end{aligned}$$

and analogously

$$\|\alpha\|_{\Xi_c,2,p,1} \leq c\left(\left\|(d_{A_c}^* d_{A_c} + d_{A_c}d_{A_c}^* - \nabla_t^{\Psi_c}\nabla_t^{\Psi_c})\alpha\right\|_{L^p} + \|\alpha\|_{\Xi_c,1,p,1}\right).$$

Next, we define a 0-form $\bar{\psi} := \psi(x, \varepsilon t)$, a 1-form $\bar{\alpha} := \alpha(x, \varepsilon t)$ and the connection $\bar{A}(x,t) + \bar{\Psi}(x,t)dt = A(x, \varepsilon t) + \varepsilon\Psi(x, \varepsilon t)dt$, then

$$\begin{aligned}
\|\psi\|_{\Xi_c,2,p,\varepsilon} &= \varepsilon^{\frac{1}{p}}\|\bar{\psi}\|_{\bar{A}+\bar{\Psi}dt,2,p,1}\\
&\leq c\varepsilon^{\frac{1}{p}}\left(\left\|d_{\bar{A}}^* d_{\bar{A}}\bar{\psi} - \nabla_t^{\bar{\Psi}}\nabla_t^{\bar{\Psi}}\bar{\psi}\right\|_{L^p} + \|\psi\|_{\bar{A}+\bar{\Psi}dt,1,p,1}\right)\\
&= c\left(\left\|d_{A_c}^* d_{A_c}\psi - \varepsilon^2\nabla_t^{\Psi_c}\nabla_t^{\Psi_c}\psi\right\|_{L^p} + \|\psi\|_{\Xi_c,1,p,\varepsilon}\right)
\end{aligned}$$

and, analogously,

$$
\begin{aligned}
\|\alpha\|_{\Xi_c,2,p,\varepsilon} &= \varepsilon^{\frac{1}{p}} \|\bar{\alpha}\|_{\bar{A}+\bar{\Psi}dt,2,p,1} \\
&\leq c\varepsilon^{\frac{1}{p}} \left(\left\| (d_{\bar{A}}^* d_{\bar{A}} + d_{\bar{A}} d_{\bar{A}}^* - \nabla_t^{\bar{\Psi}} \nabla_t^{\bar{\Psi}})\bar{\alpha} \right\|_{L^p} + \|\alpha\|_{\bar{A}+\bar{\Psi}dt,1,p,1} \right) \\
&= c \left(\left\| (d_{A_c}^* d_{A_c} + d_{A_c} d_{A_c}^* - \varepsilon^2 \nabla_t^{\Psi_c} \nabla_t^{\Psi_c})\alpha \right\|_{L^p} + \|\alpha\|_{\Xi_c,1,p,\varepsilon} \right)
\end{aligned}
$$

that establish the estimates of the third step. □

Step 3. For $1 < p < \infty$ there exists a constant c, such that

$$
\|\psi\|_{\tilde{\Xi},2,p,\varepsilon} \leq c \left(\left\| d_{\tilde{A}}^* d_{\tilde{A}} \psi - \varepsilon^2 \nabla_t^{\tilde{\Psi}} \nabla_t^{\tilde{\Psi}} \psi \right\|_{L^p} + \|\psi\|_{\tilde{\Xi},1,p,\varepsilon} \right) \tag{55}
$$

$$
\|\alpha\|_{\tilde{\Xi},2,p,\varepsilon} \leq c \left(\left\| \left(d_{\tilde{A}}^* d_{\tilde{A}} + d_{\tilde{A}} d_{\tilde{A}}^* - \varepsilon^2 \nabla_t^{\tilde{\Psi}} \nabla_t^{\tilde{\Psi}} \right) \alpha \right\|_{L^p} + \|\alpha\|_{\tilde{\Xi},1,p,\varepsilon} \right) \tag{56}
$$

holds for every 0-form $\psi \in W_c^{2,p}(U \times \mathbb{R}, \mathfrak{g})$ and every 1-form $\alpha \in W_c^{2,p}(U \times \mathbb{R}, T^*(U \times \mathbb{R}) \times \mathfrak{g})$ with compact support in $U \times \mathbb{R}$.

Proof The third step follows from the second step and the assumption (50). □

Step 4. We prove the theorem.

Proof We choose a finite atlas $\{V_i, \varphi_i : V_i \to \Sigma \times S^1\}_{i \in I}$ of our 3-manifold $\Sigma \times S^1$. Furthermore, we fix a partition of the unity $\{\rho_i\}_{i \in I} \subset C^\infty(\Sigma \times S^1, [0, 1])$, $\sum_{i \in I} \rho_i(x) = 1$ for every $x \in \Sigma \times S^1$ and $\mathrm{supp}(\rho_i) \subset \varphi_i(V_i)$ for any $i \in I$. Furthermore, we denote by $\Xi_i = A_i + \Psi_i dt \in \Omega(V_i, \mathfrak{g})$ the local representations of the connection $A + \Psi dt$ on V_i and by α_i the local representations of α. We choose the atlas in order that each Ξ_i satisfies the condition (50) for constant connections Ξ_i^c. Then by the last step

$$
\begin{aligned}
\|(\rho_i \circ \varphi_i)\alpha_i\|_{\Xi_i,2,p,\varepsilon} \leq c(\Xi_i) &\left\| \left(d_{A_i} d_{A_i}^* + d_{A_i} d_{A_i} - \nabla_t^{\Psi_i} \nabla_t^{\Psi_i} \right) ((\rho_i \circ \varphi_i)\alpha_i) \right\|_{L^p(U_i)} \\
&+ c(\Xi_i) \|(\rho_i \circ \varphi_i)\alpha_i\|_{\Xi_i,1,p,\varepsilon},
\end{aligned}
$$

If we sum all the estimates we obtain

$$
\begin{aligned}
\|\alpha\|_{A+\Psi dt,2,p,\varepsilon} &\leq c\|\alpha\|_{1,p,\varepsilon} + \sum_{i \in I} \|(\rho_i \circ \varphi_i)\alpha_i\|_{\Xi_i,2,p,\varepsilon} \\
&\leq \sum_{i \in I} c(\Xi_i) \left\| \left(d_{A_i} d_{A_i}^* + d_{A_i}^* d_{A_i} - \varepsilon^2 \nabla_t^{\Psi_i} \nabla_t^{\Psi_i} \right) ((\rho_i \circ \varphi_i)\alpha_i) \right\|_{L^p(U_i)} \\
&\quad + \sum_{i \in I} c(\Xi_i) \|(\rho_i \circ \varphi_i)\alpha_i\|_{\Xi_i,1,p,\varepsilon} + c\|\alpha\|_{1,p,\varepsilon} \\
&\leq c(\Xi) \Big(\left\| (d_A d_A^* + d_A^* d_A - \varepsilon^2 \nabla_t \nabla_t)\alpha \right\|_{L^p} + \|\alpha\|_{A+\Psi dt,1,p,\varepsilon} \Big).
\end{aligned}
$$

In the same way we can prove (48) and conclude the proof of the theorem. □

The next lemma allows us to estimate the non-harmonic part of a 1-form using its harmonic term and the elliptic operator $d_A d_A^* + d_A^* d_A - \varepsilon^2 \nabla_t^2$.

Lemma 3 *There are two positive constants c and ε_0 such that the following holds. For any i-form $\xi \in W^{2,p}$, $i = 0, 1$ and $0 < \varepsilon < \varepsilon_0$*

$$
\int_{S^1} \|\xi\|_{L^2(\Sigma)}^p \, dt \leq c \int_{S^1} \| -\varepsilon^2 \nabla_t^2 \xi + \Delta_A \xi \|_{L^2(\Sigma)}^p dt + c \int_{S^1} \|\pi_A(\xi)\|_{L^2(\Sigma)}^p dt. \tag{57}
$$

where $\Delta_A = d_A d_A^* + d_A^* d_A$.

Proof In this proof we denote the norm $\|\cdot\|_{L^2(\Sigma)}$ by $\|\cdot\|$. If we consider only the Laplace part of the operator, we obtain that

$$\int_{S^1} \|\xi\|^{p-2}\langle\xi, -\varepsilon^2\partial_t^2\xi + \Delta_A\xi\rangle dt = \int_{S^1} \|\xi\|^{p-2}\left(\varepsilon^2\|\partial_t\xi\|^2 + \|d_A\xi\|^2 + \|d_A^*\xi\|^2\right) dt$$
$$+ \int_{S^1} (p-2)\|\xi\|^{p-4}\langle\xi, \partial_t\xi\rangle^2 dt$$

and thus

$$\begin{aligned}
&\int_{S^1} \|\xi\|^{p-2}\left(\varepsilon^2\|\partial_t\xi\|^2 + \|d_A\xi\|^2 + \|d_A^*\xi\|^2\right) dt\\
&\leq \int_{S^1} \|\xi\|^{p-2}\langle\xi, -\varepsilon^2\partial_t^2\xi + \Delta_A\xi\rangle dt\\
&\leq \int_{S^1} \|\xi\|^{p-1}\|\varepsilon^2\partial_s\xi - \varepsilon^2\partial_t^2\xi + \Delta_A\xi\| \, dt\\
&\leq \left(\int_{S^1} \|\xi\|^p dt\right)^{\frac{p-1}{p}} \left(\int_{S^1} \|\varepsilon^2\partial_s\xi - \varepsilon^2\partial_t^2\xi + \Delta_A\xi\|^p dt\right)^{\frac{1}{p}}
\end{aligned} \tag{58}$$

where the second step follows from the Cauchy-Schwarz inequality and the fourth from the Hölder inequality. Therefore, by Lemma 13

$$\int_{S^1} \|\xi\|^p dt \leq \int_{S^1} \|\xi\|^{p-2}\left(\|d_A\xi\|^2 + \|d_A^*\xi\|^2 + \|\pi_A(\xi)\|^2\right) dt$$

and by (58) we have that

$$\begin{aligned}
&\leq \left(\int_{S^1} \|\xi\|^p dt\right)^{\frac{p-1}{p}} \left(\int_{S^1} \|-\varepsilon^2\partial_t^2\xi + \Delta_A\xi\|^p dt\right)^{\frac{1}{p}}\\
&\quad + \int_{S^1} \|\xi\|^{p-1}\|\pi_A(\xi)\| \, dt
\end{aligned}$$

in addition by the Hölder inequality

$$\begin{aligned}
&\leq \left(\int_{S^1} \|\xi\|^p dt\right)^{\frac{p-1}{p}} \left(\int_{S^1} \|-\varepsilon^2\partial_t^2\xi + \Delta_A\xi\|^p dt\right)^{\frac{1}{p}}\\
&\quad + \left(\int_{S^1} \|\xi\|^p dt\right)^{\frac{p-1}{p}} \left(\int_{S^1} \|\pi_A(\xi)\|^p dt\right)^{\frac{1}{p}};
\end{aligned}$$

thus, we can conclude that

$$\int_{S^1} \|\xi\|^p dt \le c \int_{S^1} \left(\| -\varepsilon^2 \partial_t^2 \xi + \Delta_A \xi\|^p + \|\pi_A(\xi)\|^p\right) dt.$$

and hence we finished the proof of the lemma using that $\|\Psi\|_{L^\infty} + \|\partial_t \Psi\|_{L^\infty}$ is bounded by a constant. □

Proof (*Theorem* 5) By Lemma 13, for any $\delta > 0$ there is a c_0 such that

$$\begin{aligned}
\|\alpha\|_{L^p}^p &\le \delta\left(\|d_A\alpha\|_{L^p}^p + \|d_A * \alpha\|_{L^p}^p\right) + c_0 \int_{S^1} \|\alpha\|_{L^2}^p dt \\
&\le \delta\left(\|d_A\alpha\|_{L^p}^p + \|d_A * \alpha\|_{L^p}^p\right) + c_0 c_1 \int_{S^1} \|\pi_A(\alpha)\|_{L^2}^p dt \\
&\quad + c_0 c_1 \int_{S^1} \| -\varepsilon^2 \nabla_t^2 \alpha + \Delta_A \alpha\|_{L^2}^p dt \\
&\le \delta\left(\|d_A\alpha\|_{L^p}^p + \|d_A * \alpha\|_{L^p}^p\right) + c_0 c_1 c_2 \|\pi_A(\alpha)\|_{L^p}^p \\
&\quad + c_0 c_1 c_2 \| -\varepsilon^2 \nabla_t^2 \alpha + \Delta_A \alpha\|_{L^p}^p \\
&\le \delta\left(\|d_A\alpha\|_{L^p}^p + \|d_A * \alpha\|_{L^p}^p\right) + c_0 c_1 c_2 \|\pi_A(\alpha)\|_{L^p}^p + c_4 \varepsilon^{2p} \|\alpha\|_{L^p}^p \\
&\quad + c_0 c_1 c_2 \varepsilon^{2p} \|\mathcal{D}_1^\varepsilon(\xi)\|_{L^p}^p + c_4 \varepsilon^{2p} \|\psi\|_{L^p}^p
\end{aligned}$$

where the second step follows form the Lemma 3 and the third by the Hölder's inequality with $c_2 := (\int_\Sigma \mathrm{dvol}_\Sigma)^{\frac{p-2}{p}}$. If we choose therefore δ and ε small enough we can improve the estimate of the Theorem 7 using the last estimate and we obtain (33), i.e.

$$\|\xi\|_{2,p,\varepsilon} \le c\left(\varepsilon^2 \|\mathcal{D}^\varepsilon(\xi)\|_{0,p,\varepsilon} + \|\pi_A(\alpha)\|_{L^p}\right);$$

furthermore (34) can be proved by

$$\begin{aligned}
\|(1-\pi_A)\xi\|_{2,p,\varepsilon} &\le c\varepsilon^2 \|\mathcal{D}^\varepsilon((1-\pi_A)\xi)\|_{0,p,\varepsilon} \\
&\le c\varepsilon^2 \left(\|\mathcal{D}^\varepsilon(\xi)\|_{0,p,\varepsilon} + \|-\nabla_t\nabla_t\pi_A(\alpha) - d * X_t(A)\pi_A(\alpha)\|_{L^p}\right) \\
&\quad + c\varepsilon^3 \left\| \frac{2}{\varepsilon^2} * [\pi_A(\alpha) \wedge *(\partial_t A - d_A\Psi)]\ dt \right\|_{L^p} \\
&\le c\left(\varepsilon^2 \|\mathcal{D}^\varepsilon(\xi)\|_{0,p,\varepsilon} + \varepsilon^2 \|\nabla_t\nabla_t\pi_A(\alpha)\|_{L^p} + \varepsilon \|\pi_A(\alpha)\|_{L^p}\right).
\end{aligned}$$

(35) follows from

$$\begin{aligned}
\|(1-\pi_A)\alpha\|_{2,p,\varepsilon} &\le c\varepsilon^2 \|\mathcal{D}^\varepsilon((1-\pi_A)\alpha)\|_{0,p,\varepsilon} \\
&\le c\varepsilon^2 \|\mathcal{D}_1^\varepsilon((1-\pi_A)\alpha)\|_{L^p} \\
&\quad + c\varepsilon \|2 * [(1-\pi_A)\alpha \wedge *(\partial_t A - d_A\Psi)]\|_{L^p} \\
&\le c\varepsilon^2 \|\mathcal{D}_1^\varepsilon(\xi)\|_{0,p,\varepsilon} + c\varepsilon \|(1-\pi_A)\alpha\|_{L^p} \\
&\quad + \varepsilon^2 \|-\nabla_t\nabla_t\pi_A(\alpha) - d * X_t(A)\pi_A(\alpha)\|_{L^p} \\
&\quad + \varepsilon^2 \|2[\psi, (\partial_t A - d_A\Psi)]\|_{L^p} \\
&\le c\varepsilon^2 \left(\|\mathcal{D}_1^\varepsilon(\xi)\|_{0,p,\varepsilon} + \|\nabla_t\nabla_t\pi_A(\alpha)\|_{L^p} + \|\pi_A(\alpha)\|_{L^p}\right) \\
&\quad + c\varepsilon \|(1-\pi_A)\alpha\|_{L^p} + c\varepsilon^2 \|\psi\|_{L^p},
\end{aligned}$$

indeed, if we choose ε small enough and we use (34) to estimate $c\varepsilon^2\|\psi\|_{L^p}$ we conclude

$$\begin{aligned}\|(1-\pi_A)\alpha\|_{2,p,\varepsilon} \le c\varepsilon^2 &\left(\|\mathcal{D}_1^\varepsilon(\xi)\|_{L^p} + \varepsilon^2\|\mathcal{D}_2^\varepsilon(\xi)\|_{L^p}\right)\\ &+ c\varepsilon^2\left(\|\nabla_t\nabla_t\pi_A(\alpha)\|_{L^p} + \|\pi_A(\alpha)\|_{L^p}\right)\end{aligned}$$

and the proof of the Theorem 5 is complete. □

8 Quadratic estimates

In the next chapter we will prove the existence and the uniqueness of a map $\mathcal{T}^{\varepsilon,b}$ between the perturbed geodesics and the perturbed Yang–Mills connection provided that ε is small enough; in order to do this we need the following quadratic estimates.

Lemma 4 *For any two constants $p \ge 2$ and $c_0 > 0$ there are two positive constants c and ε_0 such that for any two connections $A + \Psi dt$, $\tilde{A} + \tilde{\Psi} dt \in \mathcal{A}^{1,p}(P \times S^1)$*

$$\begin{aligned}&\left\|\left(\mathcal{D}^\varepsilon(A+\Psi dt) - \mathcal{D}^\varepsilon(\tilde{A}+\tilde{\Psi}dt)\right)(\alpha,\psi)\right\|_{0,p,\varepsilon}\\ &\le \frac{c}{\varepsilon^2}\|A-\tilde{A}+(\Psi-\tilde{\Psi})\,dt\|_{\infty,\varepsilon}\|\alpha+\psi dt\|_{1,p,\varepsilon}\\ &\quad+\frac{c}{\varepsilon^2}\|\alpha+\psi dt\|_{\infty,\varepsilon}\|A-\tilde{A}+(\Psi-\tilde{\Psi})\,dt\|_{1,p,\varepsilon}\end{aligned}\tag{59}$$

$$\begin{aligned}&\left\|\left(\mathcal{D}^\varepsilon(A+\Psi dt) - \mathcal{D}^\varepsilon(\tilde{A}+\tilde{\Psi}dt)\right)(\alpha,\psi)\right\|_{0,p,\varepsilon}\\ &\le \frac{c}{\varepsilon^2}\|\tilde{\alpha}+\tilde{\psi}dt\|_{\infty,\varepsilon}\|\alpha+\psi dt\|_{1,p,\varepsilon}\\ &\quad+\frac{c}{\varepsilon^2}\left(\|d_A\tilde{\alpha}\|_{L^\infty}+\|d_A^*\tilde{\alpha}\|_{L^\infty}+\varepsilon\|\nabla_t\tilde{\alpha}\|_{L^\infty}\right)\|\alpha+\psi dt\|_{0,p,\varepsilon}\\ &\quad+\frac{c}{\varepsilon^2}\left(\varepsilon\|d_A\tilde{\psi}\|_{L^\infty}+\varepsilon^2\|\nabla_t\tilde{\psi}\|_{L^\infty}\right)\|\alpha+\psi dt\|_{0,p,\varepsilon}\end{aligned}\tag{60}$$

hold for every $\alpha+\psi\, dt \in W^{1,p}$ and $\tilde{A}+\tilde{\Psi}\,dt = A+\Psi\,dt+\tilde{\alpha}+\tilde{\psi}\,dt$ with $\|\tilde{\alpha}+\tilde{\psi}\,dt\|_{\infty,\varepsilon} \le c_0$ and any $0<\varepsilon<\varepsilon_0$.

Proof On the one side, the difference between the two first components can be written as

$$\begin{aligned}&\left(\mathcal{D}_1^\varepsilon(A+\Psi dt) - \mathcal{D}_1^\varepsilon(\tilde{A}+\tilde{\Psi}dt)\right)(\alpha,\psi)\\ &= -\frac{1}{\varepsilon^2}*\left[\alpha\wedge*\left(d_{\tilde{A}}(A-\tilde{A})+\frac{1}{2}[(A-\tilde{A})\wedge(A-\tilde{A})]\right)\right]\\ &\quad-\frac{1}{\varepsilon^2}*[(A-\tilde{A})\wedge*[(A-\tilde{A})\wedge\alpha]]\\ &\quad+\frac{1}{\varepsilon^2}d_{\tilde{A}}^*[(A-\tilde{A})\wedge\alpha]-\frac{1}{\varepsilon^2}*[(A-\tilde{A})\wedge*d_{\tilde{A}}\alpha]\\ &\quad-2\left[\psi,\left(\nabla_t(A-\tilde{A})-d_{\tilde{A}}(\Psi-\tilde{\Psi})+[(\Psi-\tilde{\Psi}),(A-\tilde{A})]\right)\right]\\ &\quad-\left[(\Psi-\tilde{\Psi}),\left(\nabla_t\alpha+[(\Psi-\tilde{\Psi}),\alpha]\right)\right]-\nabla_t[(\Psi-\tilde{\Psi}),\alpha]\\ &\quad+\frac{1}{\varepsilon^2}\left[(A-\tilde{A})\wedge\left(d_{\tilde{A}}^*\alpha-*[(A-\tilde{A})\wedge*\alpha]\right)\right]\\ &\quad-\frac{1}{\varepsilon^2}d_{\tilde{A}}*[(A-\tilde{A})\wedge*\alpha]+d*X_t(\tilde{A})\alpha-d*X_t(A)\alpha\end{aligned}\tag{61}$$

and on the other side,

$$\begin{aligned}
&\Big(\mathcal{D}_2^\varepsilon(A+\Psi dt)-\mathcal{D}_2^\varepsilon(\tilde{A}+\tilde{\Psi}dt)\Big)(\alpha,\psi)\\
&=\frac{2}{\varepsilon^2}*\Big[\alpha\wedge*\Big(\nabla_t(A-\tilde{A})-d_{\tilde{A}}(\Psi-\tilde{\Psi})-[(A-\tilde{A}),(\Psi-\tilde{\Psi})]\Big)\Big]\\
&\quad-\frac{1}{\varepsilon^2}*\Big[(A-\tilde{A})\wedge*\Big([(A-\tilde{A}),\psi]+d_{\tilde{A}}\psi\Big)\Big]\\
&\quad+\frac{1}{\varepsilon^2}d_{\tilde{A}}^*[(A-\tilde{A})\wedge\psi]-\Big[(\Psi-\tilde{\Psi}),\Big([(\Psi-\tilde{\Psi}),\psi]+\nabla_t\psi\Big)\Big]\\
&\quad-\nabla_t[(\Psi-\tilde{\Psi}),\psi]. \qquad (62)
\end{aligned}$$

The lemma follows estimating term by term the last two identities. □

Next, we consider the expansions, for a connection $A+\Psi dt\in\mathcal{A}^{2,p}(P\times S^1)$ and a 1-form $\alpha+\psi dt\in W^{2,p}$,

$$\begin{aligned}
\mathcal{F}_1^\varepsilon(A+\alpha,\Psi+\psi)&=\mathcal{F}_1^\varepsilon(A,\Psi)+\mathcal{D}_1^\varepsilon(A,\Psi)(\alpha,\psi)+C_1(A,\Psi)(\alpha,\psi)\\
\mathcal{F}_2^\varepsilon(A+\alpha,\Psi+\psi)&=\mathcal{F}_2^\varepsilon(A,\Psi)+\mathcal{D}_2^\varepsilon(A,\Psi)(\alpha,\psi)dt+C_2(A,\Psi)(\alpha,\psi)dt
\end{aligned}$$

and we prove the following estimates for the non linear terms.

Lemma 5 *For any constants $c_0>0$, $p\ge 2$ and any reference connection $A_0+\Psi_0dt\in\mathcal{A}^{2,p}(P\times S^1)$, there are two positive constants c and ε_0 such that for $A+\Psi dt\in\mathcal{A}^{2,p}(P\times S^1)$*

$$\begin{aligned}
&\|C_1(A,\Psi)(\alpha,\psi)+C_2(A,\Psi)(\alpha,\psi)dt\|_{0,p,\varepsilon}\\
&\quad\le\frac{1}{\varepsilon^2}c\,\|\alpha+\psi dt\|_{\infty,\varepsilon}\|\alpha+\psi dt\|_{1,p,\varepsilon}\\
&\qquad+\frac{1}{\varepsilon^2}c\,\|\alpha+\psi dt\|_{\infty,\varepsilon}\|A-A_0+(\Psi-\Psi_0)dt\|_{1,p,\varepsilon}, \qquad (63)\\
&\left\|\pi_{A_0}\left(C_1(A,\Psi)(\alpha,\psi)\right)\right\|_{L^p}\le\frac{c}{\varepsilon^2}\|\alpha+\psi dt\|_{\infty,\varepsilon}\|(1-\pi_{A_0})\alpha+\psi dt\|_{1,p,\varepsilon}\\
&\qquad+c\|\alpha\|_{L^\infty}\|\alpha\|_{L^p}+\|\psi\|_{L^\infty}\|\nabla_t\pi_{A_0}(\alpha)\|_{L^p}\\
&\qquad+\frac{c}{\varepsilon^2}\|\alpha\|_{L^\infty}^2\left(\|\alpha\|_{L^p}+\|A-A_0\|_{L^p}\right)\\
&\qquad+\frac{c}{\varepsilon^2}\|\alpha\|_{L^\infty}\left(\left\|d_A^*(A-A_0)\right\|_{L^p}+\|(\Psi-\Psi_0)dt\|_{1,p,\varepsilon}\right)\\
&\qquad+\frac{c}{\varepsilon^2}\|A-A_0\|_{L^\infty}^2\|\alpha\|_{L^p}+c\|\psi\|_{L^\infty}^2\|A-A_0\|_{L^\infty}\|\Psi-\Psi_0\|_{L^p} \qquad (64)
\end{aligned}$$

for every $\alpha+\psi\,dt\in\mathrm{W}^{1,p}$ with norm $\|\alpha+\psi\,dt\|_{\infty,\varepsilon}<c_0$ and every $0<\varepsilon<\varepsilon_0$.

Proof By definition, C_1 and C_2 are

$$\begin{aligned}
C_1(A,\Psi)(\alpha,\psi)=&\ X_t(A+\alpha)-*X_t(A)-d*X(A)\alpha\\
&+\frac{1}{2\varepsilon^2}d_A^*[\alpha\wedge\alpha]+\frac{1}{\varepsilon^2}*[\alpha\wedge*(d_A\alpha+[\alpha\wedge\alpha])]\\
&+\nabla_t[\psi,\alpha]-[\psi,[\psi,\alpha]]+\frac{1}{\varepsilon^2}[\alpha,d_A^*(A-A_0)+d_A^*\alpha]\\
&+[\psi,(\nabla_t\alpha-d_A\psi)]+\frac{1}{\varepsilon^2}[\alpha,*[\alpha\wedge*(A-A_0)]]\\
&-\frac{1}{\varepsilon^2}d_A*[\alpha\wedge*(A-A_0)]
\end{aligned}$$

$$
\begin{aligned}
&- [\alpha \wedge (\nabla_t(\Psi - \Psi_0) + \nabla_t \psi + [\psi, ((\Psi - \Psi_0) + \psi)])] \\
&- d_A[\psi, ((\Psi - \Psi_0) + \psi)],
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
C_2(A, \Psi)(\alpha, \psi) = &\frac{1}{\varepsilon^2} * [\alpha \wedge *(\nabla_t \alpha - d_A \psi - [\alpha, \psi]) - \frac{1}{\varepsilon^2} d_A^*[\psi, \alpha] \\
&+ \frac{1}{\varepsilon^2}[\psi, (d_A^*(A - A_0 + \alpha) - *[\alpha \wedge *(A - A_0)])] \\
&+ \frac{1}{\varepsilon^2} \nabla_t * [\alpha \wedge *(A - A_0)] \\
&- [\psi, (\nabla_t(\Psi - \Psi_0 + \psi) + [\psi, (\Psi - \Psi_0)])] - \nabla_t[\psi, (\Psi - \Psi_0)]
\end{aligned} \tag{66}
$$

and if we estimate term by term, we have

$$
\begin{aligned}
&\|C_1(A, \Psi)(\alpha, \psi) + C_2(A, \Psi)(\alpha, \psi) dt\|_{0,p,\varepsilon} \\
&\quad \leq \frac{1}{\varepsilon^2} c\, \|\alpha + \psi dt\|_{\infty,\varepsilon} \|\alpha + \psi dt\|_{1,p,\varepsilon} \\
&\quad\quad + \frac{1}{\varepsilon^2} c\, \|\alpha + \psi dt\|_{\infty,\varepsilon} \|A - A_0 + (\Psi - \Psi_0) dt\|_{1,p,\varepsilon}.
\end{aligned}
$$

Next, we consider

$$
\begin{aligned}
\pi_{A_0} C_1(A, \Psi)(\alpha, \psi) = &\pi_{A_0} (X_t(A + \alpha) - *X_t(A) - d * X(A)\alpha) \\
&+ \pi_{A_0} \left(\frac{1}{\varepsilon^2} * [\alpha \wedge *(d_A \alpha + [\alpha \wedge \alpha])] \right) \\
&- \pi_{A_0} \left(\frac{1}{2\varepsilon^2} * [(A - A_0), *[\alpha \wedge \alpha]] + 2[\psi, \nabla_t \pi_{A_0}(\alpha)] \right) \\
&+ \pi_{A_0} \left([\nabla_t \psi, \alpha] + 2[\psi, \nabla_t (1 - \pi_{A_0})\alpha] - [\psi, [\psi, \alpha]] \right) \\
&+ \pi_{A_0} \left(\frac{1}{\varepsilon^2} [\alpha, d_A^*(A - A_0) + d_A^* \alpha] \right) \\
&+ \pi_{A_0} \left(-[\psi, d_A \psi] + \frac{1}{\varepsilon^2} [\alpha, *[\alpha \wedge *(A - A_0)]] \right) \\
&- \pi_{A_0} \left(\frac{1}{\varepsilon^2} [(A - A_0), *[\alpha \wedge *(A - A_0)]] \right) \\
&+ \pi_{A_0} (-[\alpha \wedge (\nabla_t(\Psi - \Psi_0) + \nabla_t \psi)]) \\
&+ \pi_{A_0} (-[\alpha \wedge [\psi, ((\Psi - \Psi_0) + \psi)]]) \\
&- \pi_{A_0} ([(A - A_0), [\psi, ((\Psi - \Psi_0) + \psi)]]),
\end{aligned} \tag{67}
$$

thus if we estimate all the summands we obtain (64). □

9 The map $\mathcal{T}^{\varepsilon,b}$ between the critical connections

In this section we define the map $\mathcal{T}^{\varepsilon,b}$ which relates the perturbed closed geodesics to the perturbed Yang–Mills connections and for this purpose we assume that the Jacobi operator is invertible for every geodesic. The definition will be based on the following two theorems.

Theorem 8 (Existence) *We choose a regular energy level b of E^H and $p \geq 2$. There are costants $\varepsilon_0,\ c > 0$ such that the following holds. If $\Xi^0 = A^0 + \Psi^0 dt \in \mathrm{Crit}^b_{E^H}$ is a perturbed closed geodesic and*

$$\alpha_0^\varepsilon(t) \in im\ \left(d^*_{A^0(t)} : \Omega^2(\Sigma, \mathfrak{g}_P) \to \Omega^1(\Sigma, \mathfrak{g}_P)\right)$$

is the unique solution of

$$d^*_{A^0} d_{A^0} \alpha_0^\varepsilon = \varepsilon^2 \nabla_t (\partial_t A^0 - d_{A^0} \Psi^0) + \varepsilon^2 * X_t(A^0), \tag{68}$$

then, for any positive $\varepsilon < \varepsilon_0$, there is a perturbed Yang–Mills connection $\Xi^\varepsilon \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$ which satisfies

$$d^{*_\varepsilon}_{\Xi^0}\left(\Xi^\varepsilon - \Xi^0\right) = 0, \quad \left\|\Xi^\varepsilon - \Xi^0\right\|_{2,p,\varepsilon} \leq c\varepsilon^2 \tag{69}$$

and, for $\alpha + \psi dt := \Xi^\varepsilon - \Xi^0$,

$$\left\|(1 - \pi_{A^0})(\alpha - \alpha_0^\varepsilon)\right\|_{2,p,\varepsilon} + \varepsilon \left\|\psi dt\right\|_{2,p,\varepsilon} \leq c\varepsilon^4, \tag{70}$$

$$\left\|\pi_{A^0}(\alpha)\right\|_{2,p,1} + \varepsilon \left\|\pi_{A^0}(\alpha)\right\|_{L^\infty} \leq c\varepsilon^2. \tag{71}$$

Remark 12 As we already mentioned, a similar version of the Theorem 8 was proved by Hong in [5] for $p = 2$ and we refer to [6] for a complete proof in our setting; the proof for a general p follows in the same way.

Remark 13 The operator $d^{*_\varepsilon}_{\Xi^0}$ is defined using the L^2-inner product as we explained in the Sect. 2 and thus, it does not depend on the choice of p.

Theorem 9 (Local uniqueness) *For any perturbed geodesic $\Xi^0 \in \mathrm{Crit}^b_{E^H}$ and any $c > 0$ there are an $\varepsilon_0 > 0$ and a $\delta > 0$ such that the following holds for any positive $\varepsilon < \varepsilon_0$. If Ξ^ε, $\bar{\Xi}^\varepsilon$ are two perturbed Yang–Mills connections that satisfy the condition*

$$d^{*_\varepsilon}_{\Xi^0}\left(\Xi^\varepsilon - \Xi^0\right) = d^{*_\varepsilon}_{\Xi^0}\left(\bar{\Xi}^\varepsilon - \Xi^0\right) = 0$$

and the estimates

$$\varepsilon \left\|\Xi^\varepsilon - \Xi^0\right\|_{2,p,\varepsilon} + \left\|(1 - \pi_{A^0})(\Xi^\varepsilon - \Xi^0 - \alpha_0^\varepsilon)\right\|_{1,p,\varepsilon} \leq c\varepsilon^3$$

with α_0^ε defined uniquely as in (68) and

$$\left\|\bar{\Xi}^\varepsilon - \Xi^0\right\|_{1,p,\varepsilon} + \left\|\bar{\Xi}^\varepsilon - \Xi^0\right\|_{\infty,\varepsilon} \leq \delta\varepsilon, \tag{72}$$

then $\bar{\Xi}^\varepsilon = \Xi^\varepsilon$.

If a connection $\tilde{\Xi}^\varepsilon \in \mathcal{A}(P \times S^1)$ satisfies $\left\|\tilde{\Xi}^\varepsilon - \Xi^0\right\|_{2,p,\varepsilon} \leq \delta' \varepsilon^{1+\frac{1}{p}}$, then it follows from the Sobolev embedding Theorem 4 for ε small enough, that $\tilde{\Xi}^\varepsilon$ satisfies (72) with $\delta = (1 + c_s)\delta'$, where c_s ist the constant of Theorem 4. Therefore the inequality $\left\|\tilde{\Xi}^\varepsilon - \Xi^0\right\|_{2,p,\varepsilon} \leq c\varepsilon^2$ implies (72) whenever $\varepsilon < \varepsilon_1$ and ε_1 is sufficiently small, i.e. if

$$\varepsilon_1 \leq \min\left\{\varepsilon_0, \left(\frac{\delta}{2c_s c}\right)^{\frac{1}{1-\frac{1}{p}}}\right\}$$

where ε_0 is given in Theorem 9. Thus, if we choose in the Theorem 9 ε_0 satisfying $c\varepsilon_0 + c_s c\varepsilon_0^{1-\frac{1}{p}} < \delta$ we have that, for each $0 < \varepsilon < \varepsilon_0$, in the ball $B_{c\varepsilon^2}\left(\Xi^0, \|\cdot\|_{2,p,\varepsilon}\right)$ there is a

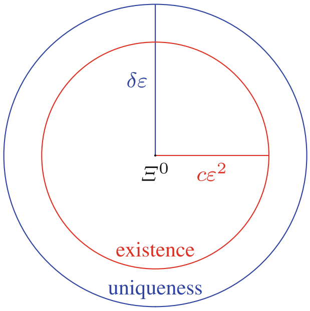


Fig. 1 Existence and uniqueness

unique perturbed Yang–Mills connection Ξ^ε which satisfies the condition $d^{*_\varepsilon}_{\Xi^0}(\Xi^\varepsilon - \Xi^0) = 0$ (Fig. 1).

Definition 1 For every regular value $b > 0$ of the energy E^H there are three positive constants ε_0, δ and c such that the assertions of the Theorems 8 and 9 hold with these constants. Shrink ε_0 such that $c\varepsilon_0 + cc_s\varepsilon_0^{1-\frac{1}{p}} < \delta$, where c_s is the constant of Sobolev Theorem 4. Theorems 8 and 9 assert that, for every $\Xi^0 \in \mathrm{Crit}^b_{E^H}$ and every ε with $0 < \varepsilon < \varepsilon_0$, there is a unique perturbed Yang–Mills connection $\Xi^\varepsilon \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$ satisfying

$$\left\| \Xi^\varepsilon - \Xi^0 \right\|_{2,p,\varepsilon} \leq c\varepsilon^2, \quad d^{*_\varepsilon}_{\Xi^0}(\Xi^\varepsilon - \Xi^0) = 0. \tag{73}$$

We define the map $\mathcal{T}^{\varepsilon,b} : \mathrm{Crit}^b_{E^H} \to \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$ by $\mathcal{T}^{\varepsilon,b}(\Xi^0) := \Xi^\varepsilon$ where $\Xi^\varepsilon \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$ is the unique Yang–Mills connection satisfying (73).

The map $\mathcal{T}^{\varepsilon,b}$ is gauge equivariant because the construction of the perturbed Yang–Mills connection in the proof of Theorem 8 is gauge equivariant, since the map $\mathcal{F}^\varepsilon$ and the operator $\mathcal{D}^\varepsilon$ are so. Furthermore, since $\mathcal{G}_0(P)$ acts free on $\mathcal{A}(P)$, the gauge group $\mathcal{G}_0(P \times S^1)$ acts freely on $\mathcal{A}(P \times S^1)$ and on the set $\mathrm{Crit}^b_{E^H}$ and thus $\mathcal{T}^{\varepsilon,b}$ defines a unique map

$$\bar{\mathcal{T}}^{\varepsilon,b} : \mathrm{Crit}^b_{E^H}/\mathcal{G}_0(P \times S^1) \to \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,b}}/\mathcal{G}_0(P \times S^1). \tag{74}$$

In addition, there is a $\gamma > 0$ which bounds from below the distance between any two different perturbed geodesics on $\mathcal{M}^g(P)$. Therefore the map $\mathcal{T}^{\varepsilon,b}$ is injective if we choose $\varepsilon < \varepsilon_1$ such that $2c\varepsilon_1^2 < \gamma$ and $\varepsilon_1 < \varepsilon_0$, where c and ε_0 are the constants in the last definition. Next, we state two useful lemmas concerning the 1-form α_0^ε; the first one follows from the regularity properties of the geodesics (cf. [5] or [6]).

Lemma 6 *For any perturbed geodesic $\Xi^0 = A^0 + \Psi^0 dt \in \mathrm{Crit}^b_{E^H}$ there is a unique 1-form $\alpha_0^\varepsilon, \alpha_0(t) \in \Omega^1(\Sigma, \mathfrak{g}_P)$, which satisfies*

$$d^*_{A^0} d_{A^0} \alpha_0^\varepsilon = \varepsilon^2 \nabla_t(\partial_t A^0 - d_{A^0}\Psi^0) + \varepsilon^2 * X_t(A^0), \quad \alpha_0^\varepsilon \in im\, d^*_{A^0}. \tag{75}$$

In addition there is a constant $c > 0$ such that

$$\|\alpha_0^\varepsilon\|_{2,p,1} + \|\alpha_0^\varepsilon\|_{L^\infty} + \|d_{A^0}\alpha_0^\varepsilon\|_{L^\infty} + \|\nabla_t \alpha_0^\varepsilon\|_{L^\infty} \leq c\varepsilon^2 \tag{76}$$

for any ε and for $\Xi_1^\varepsilon := \Xi^0 + \alpha_0^\varepsilon \in \mathcal{A}(P \times S^1)$

$$\left\| \mathcal{F}_1^\varepsilon\left(\Xi_1^\varepsilon\right) \right\|_{L^p} \leq c\varepsilon^2, \; \left\| \mathcal{F}_2^\varepsilon\left(\Xi_1^\varepsilon\right) \right\|_{L^p} \leq c. \tag{77}$$

Lemma 7 *For any perturbed geodesic $\Xi^0 = A^0 + \Psi^0 dt$ and for Ξ_1^ε defined as in Lemma 6 the following holds. There exist two constants $c > 0$ and $\varepsilon_0 > 0$ such that*

$$\begin{aligned}
&\|\pi_{A^0}(\alpha)\|_{L^p} + \|\nabla_t \pi_{A^0}(\alpha)\|_{L^p} + \|\nabla_t^2 \pi_{A^0}(\alpha)\|_{L^p} \\
&\quad \le c\varepsilon \left\|\mathcal{D}^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\right\|_{0,p,\varepsilon} + c\left\|\pi_{A^0}\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\right\|_{0,p,\varepsilon}, \qquad (78)\\
&\|\alpha - \pi_{A^0}(\alpha) + \psi\, dt\|_{2,p,\varepsilon} \\
&\quad \le c\varepsilon^2 \left\|\mathcal{D}^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\right\|_{0,p,\varepsilon} + c\varepsilon\left\|\pi_{A^0}\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\right\|_{0,p,\varepsilon}, \qquad (79)\\
&\|\alpha - \pi_{A^0}(\alpha)\|_{2,p,\varepsilon} \le c\varepsilon^2 \left\|\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\right\|_{L^p} + c\varepsilon^4 \left\|\mathcal{D}_2^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\right\|_{L^p}, \qquad (80)
\end{aligned}$$

for every $\alpha + \psi\, dt \in \mathrm{W}^{2,p}$ and any positive $\varepsilon < \varepsilon_0$.

Proof On the one side by the quadratic estimate (60)

$$\begin{aligned}
&\|\mathcal{D}^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi) - \mathcal{D}^\varepsilon(\Xi^0)(\alpha, \psi)\|_{0,p,\varepsilon} \\
&\quad \le c\varepsilon^{-2}\left(\|\alpha_0^\varepsilon\|_{L^\infty} + \|d_{A^0}\alpha_0^\varepsilon\|_{L^\infty} + \varepsilon\|\nabla_t\alpha_0^\varepsilon\|_{L^\infty}\right)\|\alpha + \psi\, dt\|_{1,p,\varepsilon} \\
&\quad \le c\|\alpha + \psi\, dt\|_{1,p,\varepsilon}. \qquad (81)
\end{aligned}$$

where the last estimate follows from (76). On the other side, we remark that the ω defined by (15) is exactly $\frac{1}{\varepsilon^2} d_{A^0}\alpha_0^\varepsilon$ and thus for the harmonic part we obtain

$$\begin{aligned}
&\pi_{A^0}\left(\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi) - \left(\mathcal{D}_1^\varepsilon(\Xi^0)(\alpha, \psi) - \frac{1}{\varepsilon^2} * [\alpha \wedge * d_{A^0}\alpha_0^\varepsilon]\right)\right) \\
&\quad = \pi_{A^0}\Big(-\frac{1}{\varepsilon^2} * \left[\alpha \wedge *\frac{1}{2}[\alpha_0^\varepsilon \wedge \alpha_0^\varepsilon]\right] - \frac{1}{\varepsilon^2} * [\alpha_0^\varepsilon \wedge *[\alpha_0^\varepsilon \wedge \alpha]] - 2\left[\psi, \nabla_t\alpha_0^\varepsilon\right] \\
&\qquad -\frac{1}{\varepsilon^2} * \left[\alpha_0^\varepsilon \wedge * d_{A^0}\alpha\right] + \frac{1}{\varepsilon^2}\left[\alpha_0^\varepsilon \wedge \left(d_{A^0}^*\alpha - *\left[\alpha_0^\varepsilon \wedge *\alpha\right]\right)\right]\Big)
\end{aligned}$$

and hence

$$\begin{aligned}
&\left\|\pi_{A^0}\left(\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi) - \mathcal{D}_1^\varepsilon(\Xi^0)(\alpha, \psi) + \frac{1}{\varepsilon^2} * [\alpha \wedge * d_{A^0}\alpha_0]\right)\right\|_{0,p,\varepsilon} \\
&\quad \le \frac{c}{\varepsilon^2}\|\alpha_0^\varepsilon\|_{L^\infty}^2\|\alpha\|_{L^p} + \frac{c}{\varepsilon^2}\left(\|\alpha_0^\varepsilon\|_{L^\infty} + \varepsilon\|\nabla_t\alpha_0^\varepsilon\|_{L^\infty}\right)\|(1 - \pi_{A^0})\alpha + \psi\, dt\|_{1,p,\varepsilon} \\
&\quad \le c\varepsilon^2\|\pi_{A^0}(\alpha)\|_{L^p} + c\|(1 - \pi_{A^0})\alpha + \psi\, dt\|_{1,p,\varepsilon}. \qquad (82)
\end{aligned}$$

By the Lemma 6 we have

$$\begin{aligned}
&\left\|\left(1 - \pi_{A^0}\right)\alpha + \psi dt\right\|_{2,p,\varepsilon} + \varepsilon\left\|\pi_{A^0}(\alpha)\right\|_{2,p,1} \\
&\quad \le c\varepsilon^2\|\mathcal{D}^\varepsilon(\Xi^0)(\alpha, \psi)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_{A^0}\left(\mathcal{D}_1^\varepsilon(\Xi^0)(\alpha, \psi) + *[\alpha \wedge *\omega]\right)\|_{L^p} \\
&\quad \le c\varepsilon^2\|\mathcal{D}^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_{A^0}\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha, \psi)\|_{0,p,\varepsilon} \\
&\qquad + c\varepsilon\|\alpha - \pi_{A^0}\alpha + \psi\, dt\|_{1,p,\varepsilon} + c\varepsilon^2\|\pi_{A^0}\alpha\|_{1,p,\varepsilon}.
\end{aligned}$$

where the second inequality follows from (81) and (82). Therefore (83) implies the first and the second estimate of the lemma choosing ε sufficiently small. The third estimates follows combining (40), (81), (82) with the first two inequality of the lemma:

$$\begin{aligned}\left\| \left(1-\pi_{A^0}\right)\alpha\right\|_{2,p,\varepsilon} &\le c\varepsilon^2\|\mathcal{D}^\varepsilon(\Xi^0)(\alpha,\psi)\|_{L^p} + c\varepsilon^4\|\mathcal{D}^\varepsilon(\Xi^0)(\alpha,\psi)\|_{L^p}\\ &\quad + c\varepsilon^2\|\pi_{A^0}\left(\mathcal{D}_1^\varepsilon(\Xi^0)(\alpha,\psi)+*[\alpha\wedge *\omega]\right)\|_{L^p}\\ &\le c\varepsilon^2\|\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha,\psi)\|_{L^p} + c\varepsilon^4\|\mathcal{D}_2^\varepsilon(\Xi_1^\varepsilon)(\alpha,\psi)\|_{L^p}\\ &\quad + c\varepsilon^2\|\alpha+\psi\, dt\|_{1,p,\varepsilon}\\ &\le c\varepsilon^2\|\mathcal{D}_1^\varepsilon(\Xi_1^\varepsilon)(\alpha,\psi)\|_{L^p} + c\varepsilon^4\|\mathcal{D}_2^\varepsilon(\Xi_1^\varepsilon)(\alpha,\psi)\|_{L^p}\end{aligned}\tag{83}$$

which is our claim. □

Next, we prove the local uniqueness theorem.

Proof (*Theorem 9*) Since Ξ^0 is a geodesic, by Lemma 6 we can define a connection $\Xi_1^\varepsilon = \Xi^0 + \alpha_0^\varepsilon$ such that $\|\alpha_0^\varepsilon\|_{2,p,1} + \|d_{A_0}\alpha_0^\varepsilon\|_{L^\infty} + \varepsilon\|\nabla_t\alpha_0^\varepsilon\|_{L^\infty} + \|\alpha_0^\varepsilon\|_{L^\infty} \le c\varepsilon^2$ and

$$\|\mathcal{F}_1^\varepsilon(\Xi_1^\varepsilon)\|_{L^p} \le c\varepsilon^2, \quad \|\mathcal{F}_2^\varepsilon(\Xi_1^\varepsilon)\|_{L^p} \le c.\tag{84}$$

Therefore we have, for $\bar\Xi^\varepsilon - \Xi_1^\varepsilon =: \alpha^\varepsilon + \psi^\varepsilon\, dt$ and $c\varepsilon < \delta$,

$$\|\bar\Xi^\varepsilon - \Xi_1^\varepsilon\|_{1,p,\varepsilon} + \|\bar\Xi^\varepsilon - \Xi_1^\varepsilon\|_{\infty,\varepsilon} \le 2\delta\varepsilon\tag{85}$$

and for $i = 1, 2$, since $\bar\Xi^\varepsilon$ is a Yang–Mills connection which satisfies $d_{\Xi^0}^{*_\varepsilon}\left(\bar\Xi^\varepsilon - \Xi^0\right) = 0$, and thus $\mathcal{F}^\varepsilon(\bar\Xi^\varepsilon) = 0$,

$$\mathcal{D}_i^\varepsilon(\Xi_1^\varepsilon)(\bar\Xi^\varepsilon - \Xi_1^\varepsilon) = -C_i^\varepsilon(\Xi_1^\varepsilon)(\bar\Xi^\varepsilon - \Xi_1^\varepsilon) - \mathcal{F}_i^\varepsilon(\Xi_1^\varepsilon).\tag{86}$$

By Lemma 7 we get

$$\begin{aligned}&\|(1-\pi_{A^0})\alpha^\varepsilon + \psi^\varepsilon\, dt\|_{2,p,\varepsilon} + \varepsilon\|\pi_{A^0}(\alpha^\varepsilon)\|_{L^p} + \varepsilon\|\nabla_t\pi_{A^0}(\alpha^\varepsilon)\|_{L^p}\\ &\quad\le c\left(\varepsilon^2\|\mathcal{D}^\varepsilon(\Xi_1^\varepsilon)(\bar\Xi^\varepsilon - \Xi_1^\varepsilon)\|_{0,p,\varepsilon} + \varepsilon\|\pi_{A^0}(\mathcal{D}^\varepsilon(\Xi_1^\varepsilon)(\bar\Xi^\varepsilon - \Xi_1^\varepsilon))\|_{0,p,\varepsilon}\right)\\ &\quad\le c\varepsilon^2\|C^\varepsilon(\Xi_1^\varepsilon(\bar\Xi^\varepsilon - \Xi_1^\varepsilon)\|_{0,p,\varepsilon} + c\varepsilon^2\|\mathcal{F}^\varepsilon(\Xi_1^\varepsilon)\|_{0,p,\varepsilon}\\ &\qquad + c\varepsilon\|\pi_{A^0}(C_1^\varepsilon(\Xi_1^\varepsilon)(\bar\Xi^\varepsilon - \Xi_1^\varepsilon))\|_{0,p,\varepsilon} + c\varepsilon\|\pi_{A^0}(\mathcal{F}_1^\varepsilon(\Xi_1^\varepsilon))\|_{0,p,\varepsilon}\\ &\quad\le c\varepsilon^3 + c\delta\|(1-\pi_{A^0})\alpha^\varepsilon + \psi^\varepsilon\, dt\|_{1,p,\varepsilon}\\ &\qquad + c\delta\left(\varepsilon\|\pi_{A^0}(\alpha^\varepsilon)\|_{L^p} + \varepsilon\|\pi_{A^0}(\alpha^\varepsilon)\|_{L^p}\right)\end{aligned}$$

where in the second step we use (86) and the third step follows from Lemma 5 and the estimate of the curvatures (84). Thus we proved the estimates $\|\bar\Xi^\varepsilon - \Xi_1^\varepsilon\|_{2,p,\varepsilon} \le c\varepsilon^2$ and hence $\|\bar\Xi^\varepsilon - \Xi^\varepsilon\|_{2,p,\varepsilon} \le c\varepsilon^2$. Since Ξ^ε satisfies $\mathcal{F}^\varepsilon(\Xi^\varepsilon) = 0$ by the assumptions, we can write

$$\begin{aligned}\mathcal{D}_i^\varepsilon(\Xi_1^\varepsilon)(\bar\Xi^\varepsilon - \Xi^\varepsilon) &= \left(\mathcal{D}_i^\varepsilon(\Xi^\varepsilon) + \left(\mathcal{D}_i^\varepsilon(\Xi_1^\varepsilon) - \mathcal{D}_i^\varepsilon(\Xi^\varepsilon)\right)\right)(\bar\Xi^\varepsilon - \Xi^\varepsilon)\\ &= -C_i^\varepsilon(\Xi^\varepsilon)(\bar\Xi^\varepsilon - \Xi^\varepsilon) + \left(\mathcal{D}_i^\varepsilon(\Xi_1^\varepsilon) - \mathcal{D}_i^\varepsilon(\Xi^\varepsilon)\right)(\bar\Xi^\varepsilon - \Xi^\varepsilon)\end{aligned}$$

and by the quadratic estimates of the Sect. 8

$$\begin{aligned}&\varepsilon^2\|C^\varepsilon(\Xi_1^\varepsilon)(\Xi^\varepsilon - \Xi_1^\varepsilon)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_{A^0}(C_1^\varepsilon(\Xi_1^\varepsilon)(\bar\Xi^\varepsilon - \Xi_1^\varepsilon))\|_{0,p,\varepsilon}\\ &\quad\le c\varepsilon^{1-\frac{1}{p}}\|(\bar\Xi^\varepsilon - \Xi^\varepsilon) - \pi_{A^0}(\bar\Xi^\varepsilon - \Xi^\varepsilon)\|_{1,p,\varepsilon}\\ &\qquad + c\varepsilon^{2-\frac{1}{p}}\|\nabla_t\pi_{A^0}(\bar\Xi^\varepsilon - \Xi^\varepsilon)\|_{0,p,\varepsilon} + c\varepsilon^{1-\frac{1}{p}}\|\pi_{A^0}(\bar\Xi^\varepsilon - \Xi^\varepsilon)\|_{0,p,\varepsilon},\end{aligned}$$

$$
\begin{aligned}
&\varepsilon^2 \|(\mathcal{D}_i^\varepsilon(\Xi_1^\varepsilon) - \mathcal{D}_i^\varepsilon(\Xi^\varepsilon))(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{0,p,\varepsilon} \\
&\quad + c\varepsilon \|\pi_{A^0}\big(\big(\mathcal{D}_i^\varepsilon(\Xi_1^\varepsilon) - \mathcal{D}_i^\varepsilon(\Xi^\varepsilon)\big)\,(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\big)\|_{0,p,\varepsilon} \\
&\le c\varepsilon^{1-\frac{1}{p}} \|(\bar{\Xi}^\varepsilon - \Xi^\varepsilon) - \pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{1,p,\varepsilon} \\
&\quad + c\varepsilon^{1-\frac{1}{p}} \|\nabla_t \pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon) \wedge dt\|_{0,p,\varepsilon} + c\varepsilon^{2-\frac{1}{p}} \|\pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{0,p,\varepsilon},
\end{aligned}
$$

we obtain by the Lemma 7

$$
\begin{aligned}
&\|(1-\pi_{A^0})(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{2,p,\varepsilon} + \varepsilon\|\pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{L^p} + \varepsilon\|\nabla_t \pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{2,p,1} \\
&\le c\varepsilon^2 \|\mathcal{D}^\varepsilon(\Xi_1)(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_{A^0}\mathcal{D}_1^\varepsilon(\Xi_1)(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{0,p,\varepsilon} \\
&\le c\varepsilon^{1-\frac{1}{p}} \|\bar{\Xi}^\varepsilon - \Xi^\varepsilon - \pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{2,p,\varepsilon} \\
&\quad + c\varepsilon^{2-\frac{1}{p}} \left(\|\pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{L^p} + \|\nabla_t \pi_{A^0}(\bar{\Xi}^\varepsilon - \Xi^\varepsilon)\|_{L^p}\right)
\end{aligned}
$$

and thus, $\|\bar{\Xi}^\varepsilon - \Xi^\varepsilon\|_{2,p,\varepsilon} = 0$ and hence $\bar{\Xi}^\varepsilon = \Xi^\varepsilon$ in for ε small enough which is the desired conclusion. □

The following theorem states a uniqueness property. The result is interesting, but it will not be used in the next chapters and in particular it will not enter in the proof of the surjectivity of $\mathcal{T}^{\varepsilon,b}$ on the contrary to what one might expects.

Theorem 10 (Uniqueness) *We choose $p > 3$. For every perturbed geodesic $\Xi^0 \in \mathrm{Crit}^b_{E^H}$ there are constants $\varepsilon_0, \delta_1 > 0$ such that the following holds. If $0 < \varepsilon < \varepsilon_0$ and $\tilde{\Xi}^\varepsilon \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$ is a perturbed Yang–Mills connection satisfying*

$$
\left\|\tilde{\Xi}^\varepsilon - \Xi^0\right\|_{1,p,\varepsilon} \le \delta_1 \varepsilon^{1+1/p}, \tag{87}
$$

then there is a $g \in \mathcal{G}_0^{2,p}(P \times S^1)$ such that $g^*\tilde{\Xi}^\varepsilon = \mathcal{T}^{\varepsilon,b}(\Xi^0)$.

Theorem 11 *Assume that $q \ge p > 2$ and $q > 3$. Let $\Xi^0 = A^0 + \Psi^0 dt \in \mathcal{A}^{1,p}(P \times S^1)$ be a connection flat on the fibers, i.e. $F_{A^0} = 0$. Then for every $c_0 > 0$ there exist $\delta_0 > 0, c > 0$ such that the following holds for $0 < \varepsilon \le 1$. If $\Xi = A + \Psi dt \in \mathcal{A}^{1,p}(P \times S^1)$ satisfies*

$$
\left\|d_{A^0}^*(A - A^0) - \varepsilon^2 \nabla_t^{\Psi^0}(\Psi - \Psi^0)\right\|_{L^p} \le c_0 \varepsilon^{1/p}, \quad \left\|\Xi - \Xi^0\right\|_{0,q,\varepsilon} \le \delta_0 \varepsilon^{1/q}, \tag{88}
$$

then there exists a gauge transformation $g \in \mathcal{G}_0^{2,p}$ such that $d^{*_\varepsilon}_{\Xi^0}(g^*\Xi - \Xi^0) = 0$ and

$$
\left\|g^*\Xi - \Xi\right\|_{1,p,\varepsilon} \le c\varepsilon^2 \left(1 + \varepsilon^{-1/p}\|\Xi - \Xi^0\|_{1,p,\varepsilon}\right) \left\|d^{*_\varepsilon}_{\Xi^0}(\Xi - \Xi^0)\right\|_{L^p}. \tag{89}
$$

Proof The proof is the same as that of proposition 6.2 in [3]. In fact the Theorem 11 is the 3-dimensional version of the proposition 6.2 in [3][5] which works with 4-dimensional connections. Between this two statements there are a few changes that are a consequence of the differences in the Sobolev properties (Theorem 4 above and lemma 4.1 in [3]). Therefore here we can work with $q > 3$ instead of $q > 4$ because we have a 3-dimensional manifold and we do not need the condition $qp/(q-p) > 4$; furthermore, we can replace $\varepsilon^{2/p}, \varepsilon^{-2/p}, \varepsilon^{2/q}$ by $\varepsilon^{1/p}, \varepsilon^{-1/p}, \varepsilon^{1/q}$ because in the proof of the Sobolev Theorem 4 we rescale a 1-dimensional domain instead of a 2-dimensional one. In addition, we remark that the gauge transformation

[5] The 0-form $d^{*_\varepsilon}_{\Xi^0}(\Xi - \Xi^0)$ in the cited proposition is defined by $d^*_{A^0}(A - A^0) - \varepsilon^2\nabla_t^{\Psi^0}(\Psi - \Psi^0) - \varepsilon^2\nabla_t^{\Phi^0}(\Phi - \Phi^0)$ and the norms are defined in chapter 4 of the paper.

g is an element of $\mathcal{G}_0^{2,p}(P)$ and this follows from the proof of the theorem; in fact, the gauge transformation $g \in \mathcal{G}^{2,p}(P \times S^1)$ is a limit of a sequence $\{g_i\}_{i\in\mathbb{N}} \subset \mathcal{G}^{2,p}(P \times S^1)$ defined by $g_i = \exp(\eta_0)\exp(\eta_1)\ldots\exp(\eta_i)$ where $\eta_i \in W^{2,p}(\Sigma \times S^1, \mathfrak{g}_P)$ are 0-forms. Therefore, the sequence $\{g_i\}_{i\in\mathbb{N}}$ and its limit are in the unit component of the gauge group, hence in $\mathcal{G}_0^{2,p}(P \times S^1)$, and this complete the proof. □

Remark 14 In the proof of the last theorem we can not use the local slice theorem directly, because although the operator $d^{*_\varepsilon}_{\Xi^0} d_{\Xi^0}$ is Fredholm and invertible on the complement of its kernel, the norm of its inverse depends on ε and hence we do not obtain an estimate independent on the metric and thus not independent on ε.

Proof (*Uniqueness Theorem* 10) Let $\tilde{\Xi}^\varepsilon = A^\varepsilon + \Psi^\varepsilon dt$ be a perturbed Yang–Mills connection which satisfies (87) with $\Xi^0 = A^0 + \Psi^0 dt$; then

$$\begin{aligned}
&\left\| d^*_{A^0}(A^\varepsilon - A^0) - \varepsilon^2 \nabla_t^{\Psi^0}(\Psi^\varepsilon - \Psi^0) \right\|_{L^p} \\
&\quad \le \left\| d^*_{A^0}(A^\varepsilon - A^0) \right\|_{L^p} + \varepsilon^2 \left\| \nabla_t^{\Psi^0}(\Psi^\varepsilon - \Psi^0) \right\|_{L^p} \\
&\quad \le 2 \left\| \tilde{\Xi}^\varepsilon - \Xi^0 \right\|_{1,p,\varepsilon} \le 2\delta_1 \varepsilon^{1+1/p}
\end{aligned} \tag{90}$$

and therefore the first condition of the assumption (88) of Theorem 11 is satisfied for ε sufficiently small; the second one follows if we choose $\delta_1 \varepsilon < \delta_0$ and $q = p$. Thus there exists, by Theorem 11, a gauge transformation $g \in \mathcal{G}_0^{2,p}$ such that $d^*_{\Xi^0}(g^* \tilde{\Xi}^\varepsilon - \Xi^0) = 0$ and

$$\begin{aligned}
\left\| g^* \tilde{\Xi}^\varepsilon - \tilde{\Xi}^\varepsilon \right\|_{1,p,\varepsilon} &\le c\varepsilon^2 \left(1 + \varepsilon^{-1/p} \left\| \tilde{\Xi}^\varepsilon - \Xi^0 \right\|_{1,p,\varepsilon}\right) \left\| d^{*_\varepsilon}_{\Xi^0}(\tilde{\Xi}^\varepsilon - \Xi^0) \right\|_{L^p} \\
&\le 2c \left\| d^*_{A^0}(A^\varepsilon - A^0) - \varepsilon^2 \nabla_t^{\Psi^0}(\Psi^\varepsilon - \Psi^0) \right\|_{L^p} \\
&\le 4c\delta_1 \varepsilon^{1+1/p}.
\end{aligned} \tag{91}$$

Then

$$\left\| g^* \tilde{\Xi}^\varepsilon - \Xi^0 \right\|_{1,p,\varepsilon} \le \left\| g^* \tilde{\Xi}^\varepsilon - \tilde{\Xi}^\varepsilon \right\|_{1,p,\varepsilon} + \left\| \tilde{\Xi}^\varepsilon - \Xi^0 \right\|_{1,p,\varepsilon} \le (4c\delta_1 + \delta_1)\varepsilon^{1+1/p} \tag{92}$$

and by the Sobolev embedding Theorem 4 we have also that

$$\left\| g^* \tilde{\Xi}^\varepsilon - \Xi^0 \right\|_{\infty,\varepsilon} \le c_s \varepsilon^{-1/p} \left\| g^* \tilde{\Xi}^\varepsilon - \Xi^0 \right\|_{1,p,\varepsilon} \le c_s(4c+1)\delta_1 \varepsilon, \tag{93}$$

where c_s is the constant in Theorem 4. Finally, we can apply Theorem 9 with $\delta_1 < \delta/((c_s + 1)(4c+1))$ for $\bar{\Xi}^\varepsilon = g^* \tilde{\Xi}^\varepsilon$ and $\Xi^\varepsilon = \mathcal{T}^{\varepsilon,b}(\Xi^0)$ and we can conclude that $g^* \tilde{\Xi}^\varepsilon = \mathcal{T}^{\varepsilon,b}(\Xi^0)$ which is the desired conclusion. □

10 A priori estimates for the perturbed Yang–Mills connections

In this chapter we explain some a priori estimates that we will need to prove the surjectivity of the map $\mathcal{T}^{\varepsilon,b}$ and we organize them in three theorems. First we show some $L^2(\Sigma)$-estimates for the curvature term F_A (Theorem 12), then the $L^2(\Sigma)$- and the $L^\infty(\Sigma)$-estimates for the curvature term $\partial_t A - d_A \Psi$ (Theorems 13 and 14).

Theorem 12 *We choose $p \geq 2$ and two constants $b, c_1 > 0$. Then there are two positive constants ε_0, c such that the following holds. For any perturbed Yang–Mills connection $A + \Psi dt \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$, with $0 < \varepsilon < \varepsilon_0$, which satisfies*

$$\sup_{t\in S^1} \|\partial_t A - d_A\Psi\|_{L^4(\Sigma)} \leq c_1, \tag{94}$$

we have the estimates[6]

$$\|F_A\|_{3,2,\varepsilon} \leq c\varepsilon^2, \tag{95}$$

$$\begin{aligned}\sup_{t\in S^1} \Big(&\|F_A\|_{L^2(\Sigma)} + \|F_A\|_{L^\infty(\Sigma)} + \|d_A^* F_A\|_{L^2(\Sigma)} \\ &+\|d_A d_A^* F_A\|_{L^2(\Sigma)} + \varepsilon\|\nabla_t F_A\|_{L^2(\Sigma)} + \varepsilon^2\|\nabla_t\nabla_t F_A\|_{L^2(\Sigma)}\Big) \leq c\varepsilon^{2-1/p}.\end{aligned} \tag{96}$$

Theorem 13 *We choose two constants $c_1, c_2 > 0$, an open interval $\Omega \subset \mathbb{R}$ and a compact set $K \subset \Omega$. Then there are two positive constants δ, c such that the following holds. For any perturbed Yang–Mills connection $A + \Psi dt \in \mathrm{Crit}^\infty_{\mathcal{YM}^{\varepsilon,H}}$ which satisfies*

$$\sup_{t\in\Omega} \|F_A\|_{L^2(\Sigma)} \leq \delta, \quad \sup_{t\in\Omega} \|\partial_t A - d_A\Psi\|_{L^4(\Sigma)} \leq c_1, \tag{97}$$

we have the estimates, for $B_t = \partial_t A - d_A\Psi$,

$$\sup_{t\in K} \varepsilon^2\|B_t\|^2_{L^2(\Sigma)} \leq c\int_\Omega \left(\varepsilon^2\|B_t\|^2_{L^2(\Sigma)} + \|F_A\|^2_{L^2(\Sigma)} + \varepsilon^2 c_{\dot X_t(A)}\right) dt, \tag{98}$$

$$\sup_{t\in K} \|d_A B_t\|^2_{L^2(\Sigma)} \leq c\int_\Omega \left(\|d_A B_t\|^2_{L^2(\Sigma)} + \frac{1}{\varepsilon^2}\|F_A\|^2_{L^2(\Sigma)} + \|B_t\|^2_{L^2(\Sigma)}\right) dt \tag{99}$$

where $\sqrt{c_{\dot X_t(A)}}$ is a constant which bounds the L^∞-norm of $\dot X_t(A)$. The constants c and δ depend on Ω and on K, but only on their length and on the distance between their boundaries. Furthermore, if $0 < \varepsilon < c_2$, then

$$\sup_{t\in S^1} \|d_A^* d_A B_t\|^2_{L^2(\Sigma)} \leq c\int_{S^1} \left(\varepsilon^2\|B_t\|^2_{L^2(\Sigma)} + \|F_A\|^2_{L^2(\Sigma)} + \varepsilon^2 c_{\dot X_t(A)}\right) dt. \tag{100}$$

Remark 15 The estimates (98) and (99) hold for any ε and this will play a fundamental role in the next section where we will have a sequence of perturbed Yang–Mills connections in $\mathrm{Crit}^\infty_{\mathcal{YM}^{\varepsilon_i,H}}$ with $\varepsilon_i \to \infty$.

Theorem 14 *We choose a constant $b > 0$. Then there are ε_0, $c > 0$ such that for every positive $\varepsilon < \varepsilon_0$ the following holds. If $\Xi^\varepsilon := A^\varepsilon + \Psi^\varepsilon dt \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$ is a perturbed Yang–Mills connection, then*

$$\left\|\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon\right\|_{L^\infty(\Sigma)} \leq c. \tag{101}$$

First we prove the next theorem.

Theorem 15 *We choose $\delta, b > 0$, then there is a positive constant ε_0 such that the following holds. For any perturbed Yang–Mills connection $A + \Psi dt \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon,H}}$, with $0 < \varepsilon < \varepsilon_0$,*

$$\sup_{t\in S^1} \|F_A\|_{L^\infty(\Sigma)} \leq \delta.$$

[6] The operator ∇_t is defined using Ψ.

Proof The theorem follows from the perturbed Yang–Mills equation and the Sobolev Theorem 4 in the following way. If we derive the identity

$$\frac{1}{\varepsilon^2} d_A^* F_A - \nabla_t B_t - *X_t(A) = 0$$

by d_A and ∇_t we obtain

$$\begin{aligned} 0 &= \frac{1}{\varepsilon^2} d_A d_A^* F_A - d_A \nabla_t B_t - d_A * X_t(A) \\ &= \frac{1}{\varepsilon^2} d_A d_A^* F_A - \nabla_t \nabla_t F_A + [B_t \wedge B_t] - d_A * X_t(A) \\ 0 &= \frac{1}{\varepsilon^2} \nabla_t d_A^* F_A - \nabla_t \nabla_t B_t - \nabla_t * X_t(A) \\ &= \frac{1}{\varepsilon^2} d_A^* d_A B_t - \nabla_t \nabla_t B_t - \frac{1}{\varepsilon^2} * [B_t, *F_A] - \nabla_t * X_t(A) \end{aligned}$$

and the L^2-norm of the Laplace part of the last two identities is

$$\begin{aligned} \varepsilon^4 \left\| \frac{1}{\varepsilon^2} d_A d_A^* F_A - \nabla_t \nabla_t F_A \right\|_{L^2}^2 &= \left\| d_A d_A^* F_A \right\|_{L^2}^2 + \varepsilon^4 \left\| \nabla_t \nabla_t F_A \right\|_{L^2}^2 \\ &\quad + \varepsilon^2 \left\| \nabla_t d_A^* F_A \right\|_{L^2}^2 + \varepsilon^2 \langle [B_t \wedge d_A^* F_A], \nabla_t F_A \rangle \\ &\quad + \varepsilon^2 \langle \nabla_t d_A^* F_A, *[B_t, *F_A] \rangle, \\ \varepsilon^4 \left\| \frac{1}{\varepsilon^2} d_A^* d_A B_t - \nabla_t \nabla_t B_t \right\|_{L^2}^2 &= \left\| d_A^* d_A B_t \right\|_{L^2}^2 + \varepsilon^4 \left\| \nabla_t \nabla_t B_t \right\|_{L^2}^2 \\ &\quad + \varepsilon^2 \left\| \nabla_t d_A B_t \right\|_{L^2}^2 + \varepsilon^2 \langle - * [B_t, *d_A B_t], \nabla_t B_t \rangle \\ &\quad - \varepsilon^2 \langle \nabla_t d_A B_t, [B_t, B_t] \rangle. \end{aligned}$$

Therefore we can estimate the $\| \cdot \|_{2,2,\varepsilon}$-norm of F_A and of B_t using the Hölder inequality and the Sobolev Theorem 4:

$$\begin{aligned} \frac{1}{2} \| F_A \|_{2,2,\varepsilon}^2 &\leq \varepsilon^4 \left\| \frac{1}{\varepsilon^2} d_A d_A^* F_A - \nabla_t \nabla_t F_A \right\|_{L^2}^2 + \| F_A \|_{L^2}^2 \\ &\quad + c\varepsilon \| B_t \|_{L^2} \| d_A^* F_A \|_{L^4} \| \nabla_t F_A \wedge dt \|_{0,4,\varepsilon} \\ &\quad + \delta \varepsilon^2 \left\| \nabla_t d_A^* F_A \right\|_{L^2}^2 + c\varepsilon^2 \| B_t \|_{L^2}^2 \| F_A \|_{L^\infty}^2 \\ &\leq \varepsilon^4 \| [B_t \wedge B_t] - d_A * X_t(A) \|_{L^2}^2 + \| F_A \|_{L^2}^2 \\ &\quad + c\varepsilon^{\frac{1}{2}} \| F_A \|_{2,2,\varepsilon}^2 + \delta \varepsilon^2 \left\| \nabla_t d_A^* F_A \right\|_{L^2}^2 \\ &\leq c\varepsilon^3 \| B_t \|_{L^2} \| B_t \|_{2,2,\varepsilon} + \varepsilon^4 \| d_A * X_t(A) \|_{L^2}^2 + \| F_A \|_{L^2}^2 \\ &\quad + c\varepsilon^{\frac{1}{2}} \| F_A \|_{2,2,\varepsilon}^2 + \delta \varepsilon^2 \left\| \nabla_t d_A^* F_A \right\|_{L^2}^2 \\ &\leq c\varepsilon^3 \| B_t \|_{2,2,\varepsilon} + \| F_A \|_{L^2}^2 + c\varepsilon^{\frac{1}{2}} \| F_A \|_{2,2,\varepsilon}^2 + \delta \varepsilon^2 \left\| \nabla_t d_A^* F_A \right\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\|B_t\|^2_{2,2,\varepsilon} \le\ & \varepsilon^4 \left\| \frac{1}{\varepsilon^2} d_A^* d_A B_t - \nabla_t \nabla_t B_t \right\|^2_{L^2} + \|B_t\|^2_{L^2} \\
& + c\varepsilon \|B_t\|_{L^2} \|d_A B_t\|_{L^4} \|\nabla_t B_t \wedge dt\|_{0,4,\varepsilon} \\
& + \delta\varepsilon^2 \|\nabla_t d_A B_t\|^2_{L^2} + c\varepsilon^2 \|B_t\|^2_{L^2} \|B_t\|^2_{L^\infty} \\
\le\ & \varepsilon^4 \left\| \frac{1}{\varepsilon^2} * [B_t, *F_A] + \nabla_t * X_t(A) \right\|^2_{L^2} + \|B_t\|^2_{L^2} \\
& + c\varepsilon^{\frac{1}{2}} \|B_t\|^2_{2,2,\varepsilon} + \delta\varepsilon^2 \|\nabla_t d_A B_t\|^2_{L^2} \\
\le\ & c\|F_A\|^2_{L^2} \|B_t\|^2_{L^\infty} + c\varepsilon^4 \|B_t\|^2_{L^2} + c\varepsilon^4 + \|B_t\|^2_{L^2} \\
& + c\varepsilon^{\frac{1}{2}} \|B_t\|^2_{2,2,\varepsilon} + \delta\varepsilon^2 \|\nabla_t d_A B_t\|^2_{L^2} \\
\le\ & 2\|B_t\|^2_{L^2} + c\varepsilon^4 + c\varepsilon^{\frac{1}{2}} \|B_t\|^2_{2,2,\varepsilon} + \delta\varepsilon^2 \|\nabla_t d_A B_t\|^2_{L^2}.
\end{aligned}$$

Hence we can conclude that, for ε small enough,

$$\begin{aligned}
\|B_t\|^2_{2,2,\varepsilon} &\le 4\|B_t\|^2_{L^2} + c\varepsilon^4 \le c, \\
\|F_A\|^2_{2,2,\varepsilon} &\le c\|F_A\|^2_{L^2} + \varepsilon^3 \|B_t\|^2_{L^2} + c\varepsilon^7 \le c\varepsilon^2
\end{aligned}$$

and thus, by the Sobolev Theorem 4, $\|F_A\|_{L^\infty} \le c\varepsilon^{-\frac{1}{2}} \|F_A\|^2_{2,2,\varepsilon} \le c\varepsilon^{\frac{1}{2}}$ which is the desired conclusion. □

In order to prove the Theorem 12 we need the following lemma.

Lemma 8 *We choose $R, r > 0$, $u : B_{R+r} \subset \mathbb{R} \to \mathbb{R}$ a C^2 function, $f, g : B_{R+r} \subset \mathbb{R} \to \mathbb{R}$ such that*

$$f \le g + \partial_t^2 u,\ u \ge 0,\ f \ge 0,\ g \ge 0,$$

then

$$\int_{B_R} f\, dt \le \int_{B_{R+r}} g\, dt + \frac{4}{r^2} \int_{B_{R+r} \setminus B_R} u\, dt. \tag{102}$$

Furthermore, if $g = cu$ for a positive constant c, then

$$\frac{1}{2} \sup_{B_R} u \le \left(c + \frac{4}{r}\right) \int_{B_{R+r}} u\, dt. \tag{103}$$

Proof For $B_R \subset \mathbb{R}^2$ and the Laplace operator instead of ∂_t^2 the first estimate was proved by Gaio and Salamon [4] and the second one by Dostoglou and Salamon in the lemma 7.3 of [3]. These two proofs apply also for our case. □

Proof (*Theorem* 12) In this proof we write B_t instead of $\partial_t A - d_A \Psi$ and we denote by $\|\cdot\|$ and by $\langle \cdot, \cdot \rangle$ respectively the L^2-norm and the L^2-product on Σ. In order to prove the Theorem 12 we will apply the last lemma where we choose u to be the L^2-norms on Σ of F_A, $\nabla_t F_A$, $d_A^* F_A$ and $\nabla_t \nabla_t F_A$; since the perturbed Yang–Mills are smooth provided that we choose ε sufficiently small, as we discussed in the Sect. 5, the regularity assumption of the Lemma 8 is satisfied. In addition we recall that the Bianchi identity tell us that

$$d_A B_t = \nabla_t F_A \tag{104}$$

and by the assumptions of the theorem

$$\int_0^1 \left(\frac{1}{\varepsilon^2}\|F_A\|^2 + \|B_t\|^2\right) dt \le b, \quad \sup_{t\in S^1} \|B_t\|_{L^4(\Sigma)} \le c_1. \tag{105}$$

Furthermore by the Theorem 15 we can assume that $\sup_{t\in S^1} \|F_A\|^2 \le \delta$ where δ satisfies the assumptions of the Lemmas 11 and 12 for $p = 2$, which allows us to estimate any 2-form:

$$\|\beta\|_{L^q(\Sigma)} \le c\|d_A^*\beta\|, \forall \beta \in \Omega^2(\Sigma, \mathfrak{g}_P), \quad 2 \le q < \infty. \tag{106}$$

Step 1. We prove the estimate (95).

Proof If we derive $\|F_A\|^2$ we obtain

$$\begin{aligned}
\partial_t^2\|F_A\|^2 &= 2\|\nabla_t F_A\|^2 + 2\langle\nabla_t\nabla_t F_A, F_A\rangle = 2\|\nabla_t F_A\|^2 + 2\langle\nabla_t d_A B_t, F_A\rangle \\
&= 2\|\nabla_t F_A\|^2 + 2\langle d_A\nabla_t B_t, F_A\rangle + 2\langle[B_t \wedge B_t], F_A\rangle \\
&= 2\|\nabla_t F_A\|^2 + 2\langle\nabla_t B_t, d_A^* F_A\rangle + 2\langle[B_t \wedge B_t], F_A\rangle \\
&= 2\|\nabla_t F_A\|^2 + \frac{2}{\varepsilon^2}\|d_A^* F_A\|^2 - 2\langle *X_t(A), d_A^* F_A\rangle + 2\langle[B_t \wedge B_t], F_A\rangle \\
&\ge 2\|\nabla_t F_A\|^2 + \frac{2}{\varepsilon^2}\|d_A^* F_A\|^2 - 2|\langle *X_t(A), d_A^* F_A\rangle| - \|B_t\|^2_{L^4(\Sigma)}\|F_A\| \\
&\ge 2\|\nabla_t F_A\|^2 + \frac{2}{\varepsilon^2}\|d_A^* F_A\|^2 - c\|F_A\| - c\|d_A^* F_A\|
\end{aligned} \tag{107}$$

where the second equality follows from the Bianchi identity (104), the third from the commutation formula (46), the fifth from the perturbed Yang–Mills equation (21) and the last one from (105). Thus, (106) and (107) imply that

$$\|F_A\|^2 \le c\|d_A^* F_A\|^2 + c\varepsilon^2\|\nabla_t F_A\|^2 \le c\partial_t^2(\varepsilon^2\|F_A\|^2) + \frac{c\varepsilon^4}{\delta_0} + c\delta_0\|F_A\|^2 \tag{108}$$

and hence for δ_0 sufficiently small

$$\|F_A\|^2 + \|d_A^* F_A\|^2 + \varepsilon^2\|\nabla_t F_A\|^2 \le c\partial_t^2(\varepsilon^2\|F_A\|^2) + c\varepsilon^4; \tag{109}$$

applying the Lemma 8 for (109)

$$\int_0^1 \left(\|F_A\|^2 + \varepsilon^2\|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right) dt \le c\varepsilon^4 + c\varepsilon^2 \int_0^1 \|F_A\|^2 dt \le c\varepsilon^4 \tag{110}$$

by (105). Analogously to (107) one can show that

$$\begin{aligned}
&\partial_t^2\left(\varepsilon^4\|\nabla_t F_A\|^2 + \varepsilon^2\|d_A^* F_A\|^2\right) \\
&\quad \ge \varepsilon^4\|\nabla_t\nabla_t F_A\|^2 + \varepsilon^2\|d_A^*\nabla_t F_A\|^2 + \|d_A d_A^* F_A\|^2 - c\varepsilon^4,
\end{aligned} \tag{111}$$

$$\begin{aligned}
&\partial_t^2\left(\varepsilon^6\|\nabla_t\nabla_t F_A\|^2 + \varepsilon^4\|\nabla_t F_A\|^2 + \varepsilon^2\|d_A^* F_A\|^2\right) \\
&\quad \ge \varepsilon^6\|\nabla_t\nabla_t\nabla_t F_A\|^2 + \varepsilon^4\|d_A^*\nabla_t\nabla_t F_A\|^2 \\
&\qquad + \varepsilon^4\|\nabla_t\nabla_t F_A\|^2 + \varepsilon^2\|d_A^*\nabla_t F_A\|^2 + \|d_A d_A^* F_A\|^2 - c\varepsilon^4,
\end{aligned} \tag{112}$$

Hence by the Lemma 8

$$\int_0^1 \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t d_A^* F_A\|^2 + \|d_A d_A^* F_A\|\right) dt$$
$$\leq c \int_0^1 \left(\varepsilon^2 \|\nabla_t F_A\|^2 + \varepsilon^2 \|d_A^* F_A\|^2 + \varepsilon^2 \|F_A\|^2 + c\varepsilon^4\right) dt \leq c\varepsilon^4, \tag{113}$$
$$\int_0^1 \left(\varepsilon^6 \|\nabla_t \nabla_t \nabla_t F_A\|^2 + \varepsilon^4 \|\nabla_t \nabla_t d_A^* F_A\|^2\right) dt \leq c\varepsilon^4. \tag{114}$$

and thus, $\|F_A\|_{3,2,\varepsilon} \leq c\varepsilon^2$ by (110), (113) and (114) and therefore we concluded the proof of the first step. □

Step 2. $\int_0^1 \left(\|F_A\|^{2p}_{L^2(\Sigma)} + \varepsilon^{2p} \|\nabla_t F_A\|^{2p}_{L^2(\Sigma)} + \varepsilon^{4p} \|\nabla_t \nabla_t F_A\|^{2p}_{L^2(\Sigma)}\right) dt \leq c\varepsilon^{4p}$.

Proof Using the estimates (111), (112) combined with the Lemma 11 we obtain

$$\begin{aligned}&\left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right)\\ &\quad \leq c\varepsilon^4 + c\varepsilon^2 \partial_t^2 \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right)\end{aligned}$$

and since for $f(t) = \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right)$

$$\partial_t^2 f(t)^p = \frac{p}{2} f(t)^{p-1} \partial_t^2 f(t) + \frac{p(p-1)}{4} f(t)^{p-2} (\partial_t f(t))^2 \geq \frac{p}{2} f(t)^{p-1} \partial_t^2 f(t)^2,$$

we have

$$\begin{aligned}&\left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right)^p\\ &\quad \leq c\varepsilon^4 \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \varepsilon^2 \|d_A^* F_A\|^2\right)^{p-1}\\ &\quad\quad + c\varepsilon^2 \partial_t^2 \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right)^p.\end{aligned}$$

Then, we apply the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with $q = \frac{p}{p-1}$ for the first term on the right side of the inequality for $a = c\varepsilon^4$ and $b = f(t)^{p-1}$ and hence

$$\begin{aligned}&\frac{1}{p}\left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right)^p\\ &\quad \leq c\varepsilon^{4p} + c\varepsilon^2 \partial_t^2 \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2\right)^p.\end{aligned} \tag{115}$$

Finally using the previous Lemma 8

$$\int_0^1 \left(\|d_A^* F_A\|^{2p}_{L^2(\Sigma)} + \varepsilon^{2p} \|\nabla_t F_A\|^{2p}_{L^2(\Sigma)} + \varepsilon^{4p} \|\nabla_t \nabla_t F_A\|^{2p}_{L^2(\Sigma)}\right) dt$$
$$\leq c_2 \varepsilon^{4p} + \varepsilon^2 \int_0^1 \left(\|d_A^* F_A\|^{2p}_{L^2(\Sigma)} + \varepsilon^{2p} \|\nabla_t F_A\|^{2p}_{L^2(\Sigma)} + \varepsilon^{4p} \|\nabla_t \nabla_t F_A\|^{2p}_{L^2(\Sigma)}\right) dt$$

and hence we conclude the proof of the second step choosing ε sufficiently small. □

Step 3. For any $p \geq 2$, the estimate (96) holds.

Proof The estimate (115) yields to

$$0 \leq c\varepsilon^2 \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2 + \varepsilon^{4-\frac{2}{p}}\right)^p$$
$$+\varepsilon^2 \partial_t^2 \left(\varepsilon^4 \|\nabla_t \nabla_t F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2 + \varepsilon^{4-\frac{2}{p}}\right)^p$$

and thus by the Lemma 8

$$\sup_{t \in S^1} \left(\varepsilon^2 \|d_A^* F_A\|^{2p} + \varepsilon^{2+2p} \|\nabla F_A\|^{2p} + \varepsilon^{2+4p} \|\nabla_t \nabla_t F_A\|^{2p}\right)$$
$$\leq c\varepsilon^{4p} + \varepsilon^2 \int_0^1 \left(\|d_A^* F_A\|^{2p} + \varepsilon^{2p} \|\nabla F_A\|^{2p} + \varepsilon^{4p} \|\nabla_t \nabla_t F_A\|^{2p}\right) dt \leq c\varepsilon^{4p}.$$

By the perturbed Yang–Mills equation we can also estimate $\|d_A d_A^* F_A\|$ in the following way:

$$\begin{aligned}
\|d_A d_A^* F_A\| &\leq \varepsilon^2 \|d_A \nabla_t B_t\| + c\varepsilon^2 \\
&\leq \varepsilon^2 \|\nabla_t d_A B_t\| + \varepsilon^2 \|[B_t \wedge B_t]\| + c\varepsilon^2 \\
&\leq \varepsilon^2 \|\nabla_t \nabla_t F_A\| + 4\varepsilon^2 \|B_t\|^4_{L^4(\Sigma)} + c\varepsilon^2
\end{aligned}$$

where the second inequality follows from (106) and the commutation formula (46) and the third from the Bianchi identity (104) and the Hölder inequality. By the last two estimates and by the Lemma 11 we can conclude that

$$\sup_{t \in S^1} \left(\|F_A\| + \|F_A\|_{L^\infty(\Sigma)} + \|d_A^* F_A\|\right.$$
$$\left. + \|d_A d_A^* F_A\| + \varepsilon \|\nabla_t F_A\| + \varepsilon^2 \|\nabla_t \nabla_t F_A\|\right) \leq c\varepsilon^{2-\frac{1}{p}}.$$

□

With the third step we finished also the proof of the Theorem 12. □

Proof (*Theorem* 13) During this proof we denote by $\|\cdot\|$ and by $\langle \cdot, \cdot \rangle$ respectively the L^2-norm and the inner product over Σ. We choose δ small enough to apply the Lemma 11 and hence $\|F_A\| \leq c\|d_A^* F_A\|$ holds for a constant c.

Step 1. There is a constant $c > 0$ such that

$$\sup_{t \in K} \varepsilon^2 \|B_t\|^2 \leq c \int_\Omega \left(\varepsilon^2 \|B_t\|^2 + \|F_A\|^2 + \varepsilon^2 c_{\dot{X}_t}\right) dt.$$

$$\int_K \|d_A B_t\|^2 dt \leq c \int_\Omega \left(\|F_A\|^2 + \frac{1}{\varepsilon^2} \|F_A\|^2 + \varepsilon^2 \|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t(A)}\right) dt.$$

Proof In order to prove the first step we compute $\partial_t^2\|B_t\|^2$ and then we apply the Lemma 8. By the perturbed Yang–Mills equation (21), we have tha

$$\begin{aligned}
\frac{1}{2}\partial_t^2\|B_t\|^2 &= \|\nabla_t B_t\|^2 + \langle \nabla_t\nabla_t B_t, B_t\rangle \\
&= \|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2}\langle \nabla_t d_A^* F_A, B_t\rangle - \langle \nabla_t * X_t(A), B_t\rangle \\
&= \|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2}\langle d_A^*\nabla_t F_A, B_t\rangle + \frac{1}{\varepsilon^2}\langle *[B_t, *F_A], B_t\rangle \\
&\quad -\langle d * X_t(A)B_t + \dot{X}_t(A), B_t\rangle \\
&= \|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2}\|d_A B_t\|^2 + \frac{1}{\varepsilon^2}\langle *[B_t, *F_A], B_t\rangle \\
&\quad - \langle d * X_t(A)B_t + \dot{X}_t(A), B_t\rangle.
\end{aligned}$$

where third step follows from the commutation formula (46) and the fourth from the Bianchi identity (104). Thus, using the Hölder, the Cauchy-Schwarz inequality and the Sobolev estimate $\|B_t\|_{L^4(\Sigma)} \leq c\,(\|B_t\| + \|d_A B_t\|)$, one can easily see that

$$\begin{aligned}
\partial_t^2\|B_t\|^2 &\geq \|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2}\|d_A B_t\|^2 - \frac{c}{\varepsilon^2}\|B_t\|_{L^4}(\|B_t\| + \|d_A B_t\|)\|F_A\| \\
&\quad -c\|B_t\|^2 - c\|\dot{X}_t(A)\|\cdot\|B_t\| \\
&\geq \|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2}\|d_A B_t\|^2 - \frac{c}{\varepsilon^4}\|F_A\|^2 \\
&\quad -\frac{c}{\varepsilon^2}\|F_A\|^2 - c\|B_t\|^2 - c\|\dot{X}_t(A)\|^2.
\end{aligned} \tag{116}$$

Hence using the Lemma 8 we can conclude the second estimate of the first step:

$$\begin{aligned}
&\int_{S^1}\left(\varepsilon^2\|\nabla_t B_t\|^2 + \|d_A B_t\|^2\right)dt \\
&\quad \leq c\int_{S^1}\left(\frac{1}{\varepsilon^2}\|F_A\|^2 + \varepsilon^2\|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t(A)} + c\|F_A\|^2\right)dt.
\end{aligned}$$

Since $\|F_A\|_{L^2(\Sigma)} \leq \delta$ and $\|F_A\| \leq c\|d_A^* F_A\|$, by the Theorems 11 and 12 there is a $A_1 \in \mathcal{A}^0(P)$ such that $\|A - A_1\|_{L^2} \leq c\|F_A\|_{L^2}$ and thus we can write

$$d_A * X_t(A) = d_{A_1} * X_t(A_1) + [(A - A_1)\wedge *X_t(A_1)] \tag{117}$$

where $d_{A_1} * X_t(A_1) = 0$. Therefore, by the fifth line of the computation (107)

$$\frac{1}{2}\partial_t^2\|F_A\|^2 \geq \frac{1}{4\varepsilon^2}\|d_A^* F_A\|^2 + \frac{1}{4}\|\nabla_t F_A\|^2 - c\varepsilon^2\|B_t\|^2 - c\|F_A\|^2 \tag{118}$$

and with (116) it follows that for a constant c_0 big enough

$$\frac{1}{2}\partial_t^2\left(c_0\|F_A\|^2 + \varepsilon^2\|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t}\right) \geq -c\left(c_0\|F_A\|^2 + \varepsilon^2\|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t}\right).$$

Finally by Lemma 8, we can conclude that

$$\sup_{t\in K}\varepsilon^2\|B_t\|^2 \leq c\int_{\Omega}\left(\varepsilon^2\|B_t\|^2 + \|F_A\|^2 + \varepsilon^2 c_{\dot{X}_t}\right)dt.$$

□

Step 2. There is a positive constant $c > 0$ such that

$$\sup_{t\in K} \|d_A^* d_A B_t\|^2 \leq c \int_\Omega \left(\|F_A\|^2 + \varepsilon^2 \|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t(A)} + \|d_A^* d_A B_t\|^2 \right) dt.$$

Proof Analogously to the previous step we need to compute $\frac{1}{2}\partial_t^2 \|d_A^* d_A B_t\|^2$:

$$\frac{1}{2}\partial_t^2 \; ||d_A^* d_A B_t||^2 = ||\nabla_t d_A^* d_A B_t||^2 + \langle \nabla_t \nabla_t d_A^* d_A B_t, d_A^* d_A B_t\rangle$$

by the commutation formula (46) and the Yang–Mills equation (21) we have

$$\begin{aligned} = \; & ||\nabla_t d_A^* d_A B_t||^2 + \frac{1}{\varepsilon^2}\langle \nabla_t d_A^* d_A d_A^* F_A, d_A^* d_A B_t\rangle \\ & -\langle \nabla_t d_A^* d_A * X_t(A), d_A^* d_A B_t\rangle \\ & +\langle \nabla_t \left(- * [B_t \wedge, * d_A B_t] + d_A^*[B_t \wedge B_t]\right), d_A^* d_A B_t\rangle \end{aligned}$$

and applying one more time t(46)

$$\begin{aligned} = \; & ||\nabla_t d_A^* d_A B_t||^2 + \frac{1}{\varepsilon^2}\langle d_A^* d_A d_A^* \nabla_t F_A, d_A^* d_A B_t\rangle \\ & +\frac{1}{\varepsilon^2}\langle - * [B_t \wedge * d_A d_A^* F_A] + d_A^*[B_t \wedge d_A^* F_A], d_A^* d_A B_t\rangle \\ & -\frac{1}{\varepsilon^2}\langle d_A * [B_t, *F_A], d_A d_A^* d_A B_t\rangle - \langle d_A * \nabla_t X_t(A), d_A d_A^* d_A B_t\rangle \\ & -\langle d_A^*[B_t \wedge *X_t(A)] - *[B_t \wedge * d_A * X_t(A)], d_A^* d_A B_t\rangle \\ & +\langle \nabla_t \left(- * [B_t \wedge, * d_A B_t] + d_A^*[B_t \wedge B_t]\right), d_A^* d_A B_t\rangle \end{aligned}$$

finally, by the Bianchi identity (104) and the perturbed Yang–Mills equation (21) we can conclude that

$$\begin{aligned} = \; & ||\nabla_t d_A^* d_A B_t||^2 + \frac{1}{\varepsilon^2}||d_A d_A^* d_A B_t||^2 \\ & -\langle [*B_t \wedge * d_A(\nabla_t B_t + *X_t(A))], d_A^* d_A B_t\rangle + \frac{1}{\varepsilon^2}\langle d_A^*[B_t \wedge d_A^* F_A], d_A^* d_A B_t\rangle \\ & -\frac{1}{\varepsilon^2}\langle d_A * [B_t, *F_A], d_A d_A^* d_A B_t\rangle - \langle d_A * \nabla_t X_t(A), d_A d_A^* d_A B_t\rangle \\ & -\langle d_A^*[B_t \wedge *X_t(A)] - *[B_t \wedge * d_A * X_t(A)], d_A^* d_A B_t\rangle \\ & +\langle \nabla_t \left(- * [B_t \wedge, * d_A B_t] + d_A^*[B_t \wedge B_t]\right), d_A^* d_A B_t\rangle; \end{aligned}$$

The last computation implies

$$\partial_t^2 \|d_A^* d_A B_t\|^2 \geq -c\varepsilon^2 \|B_t\|^2 - c\frac{1}{\varepsilon^2}\|d_A^* F_A\|^2 - c\varepsilon^2 \|\nabla_t B_t\|^2 - c\|d_A^* d_A B_t\|^2.$$

Therefore combining (116), (118) and (119)

$$\begin{aligned} \partial_t^2 \left(\|d_A^* d_A B_t\|^2 + c_0 \|F_A\|^2 + c_0 \varepsilon^2 \|B_t\| \right) \geq & - c\varepsilon^2 \|B_t\|^2 - c\|F_A\|^2 \\ & - c\|d_A^* d_A B_t\|^2 - \varepsilon^2 c_{\dot{X}_t} \end{aligned}$$

and hence we conclude by the Lemma 8 that

$$\sup_{t\in K}\|d_A^*d_AB_t\|^2\le c\int\limits_{\Omega}\left(\|F_A\|^2+\varepsilon^2\|B_t\|^2+\varepsilon^2c_{\dot X_t(A)}+\|d_A^*d_AB_t\|^2\right)dt.$$

□

Step 3. There is a constant $c>0$ such that

$$\sup_{t\in K}\|d_AB_t\|\le c\int\limits_{\Omega}\left(\|d_AB_t\|^2+\frac{1}{\varepsilon^2}\|F_A\|^2+\|B_t\|^2\right)dt$$

and if $0<\varepsilon<c_2$, then

$$\int\limits_{S^1}\|d_A^*d_AB_t\|^2dt\le c\varepsilon^2\int\limits_{S^1}\left(\|F_A\|^2+\varepsilon^2\|B_t\|^2+\varepsilon^2c_{\dot X_t(A)}\right)dt.$$

Proof Like in the previous steps we will prove this one using the Lemma 8 and therefore we need to compute $\frac{1}{2}\partial_t^2||d_AB_t||^2$. We consider

$$\frac{1}{2}\partial_t^2||d_AB_t||^2=||\nabla_td_AB_t||^2+\langle\nabla_t\nabla_td_AB_t,d_AB_t\rangle$$

using the commutation formula (46) and the Yang–Mills flow equation (21), we have

$$\begin{aligned}&=||\nabla_td_AB_t||^2+\frac{1}{\varepsilon^2}\langle\nabla_td_Ad_A^*F_A,d_AB_t\rangle\\&\quad-\langle\nabla_td_A*X_t(A),d_AB_t\rangle+\langle\nabla_t[B_t\wedge B_t],d_AB_t\rangle\end{aligned}$$

by the commutation formula (46)

$$\begin{aligned}&=||\nabla_td_AB_t||^2+\frac{1}{\varepsilon^2}\langle d_Ad_A^*\nabla_tF_A,d_AB_t\rangle\\&\quad+\frac{1}{\varepsilon^2}\langle[B_t\wedge d_A^*F_A],d_AB_t\rangle-\frac{1}{\varepsilon^2}\langle*[B_t,*F_A],d_A^*d_AB_t\rangle\\&\quad-\langle\nabla_td_A*X_t(A),d_AB_t\rangle+\langle\nabla_t[B_t\wedge B_t],d_AB_t\rangle\end{aligned}$$

next, the Bianchi identity (104) yields to

$$\begin{aligned}&=||\nabla_td_AB_t||^2+\frac{1}{\varepsilon^2}||d_A^*d_AB_t||^2+\frac{1}{\varepsilon^2}\langle[B_t\wedge d_A^*F_A],d_AB_t\rangle\\&\quad-\frac{1}{\varepsilon^2}\langle*[B_t,*F_A],d_A^*d_AB_t\rangle-\langle\nabla_td_A*X_t(A),d_AB_t\rangle\\&\quad+\langle2[\nabla_tB_t\wedge B_t],d_AB_t\rangle\end{aligned}$$

and thus

$$\begin{aligned}\frac{1}{2}\partial_t^2\|d_AB_t\|^2\ge{}&\frac{1}{2}\|\nabla_td_AB_t\|^2+\frac{1}{2\varepsilon^2}\|d_A^*d_AB_t\|^2-\frac{c}{\varepsilon^2}\|d_A^*F_A\|^2\\&-c\varepsilon^2\|B_t\|^2-c\varepsilon^2\|\dot X_t(A)\|_{L^\infty}^2-c\varepsilon^2\|\nabla_tB_t\|^2.\end{aligned}\tag{119}$$

and

$$\frac{1}{2}\partial_t^2\|d_A B_t\|^2 \geq \frac{1}{2}\|\nabla_t d_A B_t\|^2 + \frac{1}{2\varepsilon^2}\|d_A^* d_A B_t\|^2 - \frac{c}{\varepsilon^4}\|d_A^* F_A\|^2 \\ - c\,\|d_A B_t\|^2 - c\|B_t\|^2 - \frac{c}{\varepsilon^2}\|F_A\|^2. \tag{120}$$

Therefore, (120) combined with (116) yields to

$$\partial_t^2\left(\|d_A B_t\|^2 + c_0\frac{1}{\varepsilon^2}\|F_A\|^2 + c_0\|B_t\|^2\right) \\ \geq -c\|B_t\|^2 - c\|\dot{X}_t(A)\|_{L^\infty} - \frac{c}{\varepsilon^2}\|F_A\|^2 - c\|d_A B_t\|^2 \tag{121}$$

where we use that

$$\partial_t^2\|F_A\|^2 \geq -c\varepsilon^2\|B_t\|^2 - c\varepsilon^2\|d_A B_t\|^2$$

by the fifth line of (107). The Lemma 8 applyed the last estimate give us

$$\sup_{t\in K}\|d_A B_t\| \leq c\int_\Omega\left(\|d_A B_t\|^2 + \frac{1}{\varepsilon^2}\|F_A\|^2 + \|B_t\|^2\right)dt.$$

The estimate (119) combined with (116) yields to

$$\partial_t^2\left(\|d_A B_t\|^2 + c_0\|F_A\|^2 + c_0\varepsilon^2\|B_t\|^2\right) \\ \geq \|\nabla_t d_A B_t\|^2 + \frac{1}{\varepsilon^2}\|d_A^* d_A B_t\|^2 - c\varepsilon^2\|B_t\|^2 - c\varepsilon^2\|\dot{X}_t(A)\|_{L^\infty} - c\|F_A\|^2 \tag{122}$$

for a constant c_0 big enough.

Hence, if $0 < \varepsilon < c_2$, by Lemma 8 we have

$$\begin{aligned}\int_{S^1}&\left(\varepsilon^2\|\nabla_t d_A B_t\|^2 + \|d_A^* d_A B_t\|^2\right)dt\\ &\leq c\varepsilon^2\int_{S^1}\left(\|d_A B_t\|^2 + \|F_A\|^2 + \varepsilon^2\|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t(A)}\right)dt\\ &\leq c\varepsilon^2\int_{S^1}\left(\|d_A^* d_A B_t\|^2 + \|F_A\|^2 + \varepsilon^2\|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t(A)}\right)dt\\ &\leq c\varepsilon^2\int_{S^1}\left(\varepsilon^2\|d_A B_t\|^2 + \|F_A\|^2 + \varepsilon^2\|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t(A)}\right)dt\\ &\leq c\varepsilon^2\int_{S^1}\left(\|F_A\|^2 + \varepsilon^2\|B_t\|^2 + \varepsilon^2 c_{\dot{X}_t(A)}\right)dt\end{aligned} \tag{123}$$

where the second estimate follows from the Lemma 11, the third inequality follows from the first one and the first step implies the last estimate. □

The estimate (100) follows combining the second and the third step; hence, we finished the proof of the Theorem 13. □

Proof (*Theorem* 14) If we prove that $\|\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon\|_{L^4(\Sigma)}$ is uniformly bounded by a constant, then by the Theorem 13 and the Sobolev estimate it follows that

$$\|\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon\|_{L^\infty(\Sigma)} \le \|\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon\|_{L^4(\Sigma)} + \|d^*_{A^\varepsilon} d_{A^\varepsilon}(\partial_t A^\varepsilon - d_{A^\varepsilon}\Psi^\varepsilon)\|_{L^2(\Sigma)} \le c$$

and hence (101) is satisfied for ε sufficiently small. We prove the theorem by an indirect argument assuming that there is a sequence $\{\Xi^{\varepsilon_\nu} = A^{\varepsilon_\nu} + \Psi^{\varepsilon_\nu} dt\}_{\nu\in\mathbb{N}}$, $\varepsilon_\nu \to 0$, of perturbed Yang–Mills connections that satisfies $\mathcal{YM}^{\varepsilon_\nu,H}(\Xi^{\varepsilon_\nu}) \le b$ and

$$m_\nu := \sup_{t\in S^1} \left\|\partial_t A^{\varepsilon_\nu} - d_{A^{\varepsilon_\nu}}\Psi^{\varepsilon_\nu}\right\|_{L^4(\Sigma)} \to \infty.$$

In addition we assume that there is a convergent sequence $t_\nu \to t^\infty$ in S^1 such that

$$\left\|\partial_t A^{\varepsilon_\nu}(t_\nu) - d_{A^{\varepsilon_\nu}(t_\nu)}\Psi^{\varepsilon_\nu}(t_\nu)\right\|_{L^4(\Sigma)} = m_\nu. \tag{124}$$

We need to check three cases that depend from the behavior of the sequence $\varepsilon_\nu m_\nu$.

Case 1 $\lim_{\nu\to\infty} \varepsilon_\nu m_\nu = 0$. We define a new sequence of connections $\bar{\Xi}^{\varepsilon_\nu} := \bar{A}^{\varepsilon_\nu} + \bar{\Psi}^{\varepsilon_\nu} dt$ by $\bar{A}^{\varepsilon_\nu}(t) := A^{\varepsilon_\nu}(t_\nu + t/m_\nu)$, and $\bar{\Psi}^{\varepsilon_\nu}(t) := \frac{1}{m_\nu}\Psi^{\varepsilon_\nu}(t_\nu + t/m_\nu)$. This sequence satisfies the perturbed Yang–Mills equations

$$\begin{aligned}\frac{1}{\varepsilon_\nu^2 m_\nu^2} d^*_{\bar{A}^{\varepsilon_\nu}} F_{\bar{A}^{\varepsilon_\nu}} &= \bar{\nabla}_t\left(\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}}\bar{\Psi}^{\varepsilon_\nu}\right) + \frac{1}{m_\nu^2} * X_{t_\nu + \frac{t}{m_\nu}}(\bar{A}^{\varepsilon_\nu}),\\ d^*_{\bar{A}^{\varepsilon_\nu}}\left(\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}}\bar{\Psi}^{\varepsilon_\nu}\right) &= 0.\end{aligned}$$

In addition, we have the following estimates for the norms for $\bar{\varepsilon}_\nu := \varepsilon_\nu m_\nu$

$$\sup_{t\in[-\frac{m_\nu}{2},\frac{m_\nu}{2}]} \left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}}\bar{\Psi}^{\varepsilon_\nu}\right\|_{L^4(\Sigma)} = \left\|\partial_t \bar{A}^{\varepsilon_\nu}(0) - d_{\bar{A}^{\varepsilon_\nu}(0)}\bar{\Psi}^{\varepsilon_\nu}(0)\right\|_{L^4(\Sigma)} = 1, \tag{125}$$

$$\begin{aligned}\frac{1}{\bar{\varepsilon}_\nu^2}\left\|F_{\bar{A}^{\varepsilon_\nu}}\right\|_{L^2}^2 &= \int_{-\frac{m_\nu}{2}}^{\frac{m_\nu}{2}} \frac{1}{\bar{\varepsilon}_\nu^2}\left\|F_{\bar{A}^{\varepsilon_\nu}}\right\|_{L^2(\Sigma)}^2 dt\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{m_\nu \varepsilon_\nu^2}\|F_{A^{\varepsilon_\nu}}\|_{L^2(\Sigma)}^2 dt \le \frac{b^2}{m_\nu},\end{aligned} \tag{126}$$

$$\begin{aligned}\left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}}\bar{\Psi}^{\varepsilon_\nu}\right\|_{L^2}^2 &= \int_{-\frac{m_\nu}{2}}^{-\frac{m_\nu}{2}} \left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}}\bar{\Psi}^{\varepsilon_\nu}\right\|_{L^2(\Sigma)}^2 dt\\ &= \int_{-\frac{1}{2}}^{-\frac{1}{2}} \frac{1}{m_\nu^2}\left\|\partial_t A^{\varepsilon_\nu} - d_{A^{\varepsilon_\nu}}\Psi^{\varepsilon_\nu}\right\|_{L^2(\Sigma)}^2 m_\nu dt \le \frac{b^2}{m_\nu}.\end{aligned} \tag{127}$$

We denote $\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}}\bar{\Psi}^{\varepsilon_\nu}$ by $\bar{B}_t^\nu$ and we remark that the L^∞-norm of $\frac{1}{m_\nu^2}\dot{X}_{t_\nu+\frac{t}{m_\nu}}(\bar{A})$ can be estimate by $\frac{c}{m_\nu^3}$ where c is a positive constant; thus, by the Sobolev estimate and the

Theorem 13 we can conclude that

$$\sup_{t\in[-\frac{m_\nu}{2},\frac{m_\nu}{2}]} \|\bar{B}_t^\nu\|^2_{L^4(\Sigma)} \le c \sup_{t\in[-\frac{m_\nu}{2},\frac{m_\nu}{2}]} \left(\|\bar{B}_t^\nu\|^2_{L^2(\Sigma)} + \|d^*_{\bar{A}^{\varepsilon_\nu}} d_{\bar{A}^{\varepsilon_\nu}} \bar{B}_t^\nu\|^2_{L^2(\Sigma)}\right)$$
$$\le c \int_{-\frac{m_\nu}{2}}^{\frac{m_\nu}{2}} \left(\|\bar{B}_t^\nu\|^2_{L^2(\Sigma)} + \frac{1}{\varepsilon_\nu^2 m_\nu^2}\|F_{\bar{A}^{\varepsilon_\nu}}\|^2_{L^2(\Sigma)} + \frac{1}{m_\nu^3}\right) dt$$
$$\le \frac{c}{m_\nu}\left(1 + \frac{1}{m_\nu} + \frac{1}{m_\nu^2}\right)$$

which converges to 0 in contradiction to (125).

Case 2 $\lim_{\nu\to\infty} \varepsilon_\nu m_\nu = c_1 > 0$. This time we choose a different rescaling to define $\bar{\Xi}^{\varepsilon_\nu} := \bar{A}^{\varepsilon_\nu} + \bar{\Psi}^{\varepsilon_\nu} dt$, i.e.

$$\bar{A}^{\varepsilon_\nu}(t) := A^{\varepsilon_\nu}(t_\nu + \varepsilon_\nu t), \quad \bar{\Psi}^{\varepsilon_\nu}(t) := \varepsilon_\nu \Psi^{\varepsilon_\nu}(t_\nu + \varepsilon_\nu t)$$

which satisfies the perturbed Yang–Mills equations

$$d^*_{\bar{A}^{\varepsilon_\nu}} F_{\bar{A}^{\varepsilon_\nu}} = \bar{\nabla}_t\left(\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right) + \varepsilon_\nu^2 * X_{t_\nu+\varepsilon_\nu t}(\bar{A}^{\varepsilon_\nu}),$$
$$d^*_{\bar{A}^{\varepsilon_\nu}}\left(\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right) = 0$$

and

$$\sup_{t\in\left[-\frac{1}{2\varepsilon_\nu},\frac{1}{2\varepsilon_\nu}\right]} \left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right\|_{L^4(\Sigma)} = \left\|\partial_t \bar{A}^{\varepsilon_\nu}(0) - d_{\bar{A}^{\varepsilon_\nu}(0)} \bar{\Psi}^{\varepsilon_\nu}(0)\right\|_{L^4(\Sigma)} \le 2c_1 \tag{128}$$

for ν sufficiently big. Furthermore, we have the estimates

$$\left\|F_{\bar{A}^{\varepsilon_\nu}}\right\|^2_{L^2} = \int_{-\frac{1}{2\varepsilon_\nu}}^{\frac{1}{2\varepsilon_\nu}} \left\|F_{\bar{A}^{\varepsilon_\nu}}\right\|^2_{L^2(\Sigma)} dt$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\varepsilon_\nu}\|F_{A^{\varepsilon_\nu}}\|^2_{L^2(\Sigma)} dt \le b\varepsilon_\nu, \tag{129}$$

$$\left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right\|^2_{L^2} = \int_{-\frac{1}{2\varepsilon_\nu}}^{\frac{1}{2\varepsilon_\nu}} \left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right\|^2_{L^2(\Sigma)} dt$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \varepsilon_\nu^2 \left\|\partial_t A^{\varepsilon_\nu} - d_{A^{\varepsilon_\nu}} \Psi^{\varepsilon_\nu}\right\|^2_{L^2(\Sigma)} \frac{1}{\varepsilon_\nu} dt \le b\varepsilon_\nu. \tag{130}$$

If we denote $\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}$ by $\bar{B}_t^\nu$ and we consider $c\varepsilon_\nu^3$ as the bound for the L^∞-norm of $\varepsilon_\nu^2 \dot{X}_{t_\nu+\varepsilon_\nu t}(\bar{A})$, then, by the Sobolev estimate and the Theorem 13 we can conclude that

$$\begin{aligned}\sup \|\bar{B}_t^\nu\|^2_{L^4(\Sigma)} &\leq c \sup \left(\|\bar{B}_t^\nu\|^2_{L^2(\Sigma)} + \|d^*_{\bar{A}^{\varepsilon_\nu}} d_{\bar{A}^{\varepsilon_\nu}} \bar{B}_t^\nu\|^2_{L^2(\Sigma)} \right) \\ &\leq c \int_{S^1} \left(\|\bar{B}_t^\nu\|^2_{L^2(\Sigma)} + \|F_{\bar{A}^{\varepsilon_\nu}}\|^2_{L^2(\Sigma)} + \varepsilon_\nu^3 \right) dt \\ &\leq c\varepsilon_\nu \left(1 + \varepsilon_\nu + \varepsilon_\nu^2\right)\end{aligned}$$

which converges to 0 in contradiction to (128).

Case 3 $\lim_{\nu\to\infty} \varepsilon_\nu m_\nu = \infty$. First, we define $\bar{\Xi}^{\varepsilon_\nu} := \bar{A}^{\varepsilon_\nu} + \bar{\Psi}^{\varepsilon_\nu} dt$ as in the first case, i.e.

$$\bar{A}^{\varepsilon_\nu}(t) := A^{\varepsilon_\nu}\left(t_\nu + \frac{t}{m_\nu}\right), \quad \bar{\Psi}^{\varepsilon_\nu}(t) := \frac{1}{m_\nu} \Psi^{\varepsilon_\nu}\left(t_\nu + \frac{t}{m_\nu}\right).$$

The new sequence satisfies then

$$\begin{aligned} d^*_{\bar{A}^{\varepsilon_\nu}} F_{\bar{A}^{\varepsilon_\nu}} &= \varepsilon_\nu^2 m_\nu^2 \bar{\nabla}_t \left(\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right) + \varepsilon_\nu^2 * X_{t_\nu + t/m_\nu}(\bar{A}^{\varepsilon_\nu}), \\ d^*_{\bar{A}^{\varepsilon_\nu}} &\left(\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right) = 0. \end{aligned} \tag{131}$$

In addition, we obtain the following estimates for any compact set $K \subset \mathbb{R}$ that

$$\sup_{t\in\left[-\frac{1}{2m_\nu}, \frac{1}{2m_\nu}\right]} \left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right\|_{L^4(\Sigma)} = \left\|\partial_t \bar{A}^{\varepsilon_\nu}(0) - d_{\bar{A}^{\varepsilon_\nu}(0)} \bar{\Psi}^{\varepsilon_\nu}(0)\right\|_{L^4(\Sigma)} = 1,$$

$$\left\|F_{\bar{A}^{\varepsilon_\nu}}\right\|^2_{L^2(\Sigma\times K)} = \int_K \left\|F_{\bar{A}^{\varepsilon_\nu}}\right\|^2_{L^2(\Sigma)} dt \leq m_\nu \int_K \|F_{A^{\varepsilon_\nu}}\|^2_{L^2(\Sigma)} dt \leq c\varepsilon_\nu^2 m_\nu, \tag{132}$$

$$\begin{aligned}\left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right\|^2_{L^2(K)} &= \int_K \left\|\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}\right\|^2_{L^2(\Sigma)} dt \\ &\leq \int_K \frac{1}{m_\nu^2} \left\|\partial_t A^{\varepsilon_\nu} - d_{A^{\varepsilon_\nu}} \Psi^{\varepsilon_\nu}\right\|^2_{L^2(\Sigma)} m_\nu dt \leq \frac{b}{m_\nu}. \end{aligned} \tag{133}$$

Analogously as in the first two cases we denote $\partial_t \bar{A}^{\varepsilon_\nu} - d_{\bar{A}^{\varepsilon_\nu}} \bar{\Psi}^{\varepsilon_\nu}$ by $\bar{B}_t^\nu$ and we consider $\frac{1}{m_\nu^3}$ as the bound for the L^∞-norm of $\frac{1}{m_\nu^2} \dot{X}_{t_\nu + \frac{t}{m_\nu}}(\bar{A})$, then, by the Sobolev estimate and the Theorem 13 we can conclude that, for a compact set K and an open set Ω with $0 \in K \subset \Omega$,

$$\begin{aligned}\sup_{t\in K} \|\bar{B}_t^\nu\|^2_{L^4(\Sigma)} &\leq c \sup_{t\in K} \left(\|\bar{B}_t^\nu\|^2_{L^2(\Sigma)} + \|d_{\bar{A}^{\varepsilon_\nu}} \bar{B}_t^\nu\|^2_{L^2(\Sigma)} \right) \\ &\leq c \int_\Omega \left(\|\bar{B}_t^\nu\|^2_{L^2(\Sigma)} + \frac{1}{\varepsilon_\nu^2 m_\nu^2} \|F_{\bar{A}^{\varepsilon_\nu}}\|^2_{L^2(\Sigma)} \right) dt \\ &\quad + c \int_\Omega \left(\frac{\varepsilon_\nu^2}{m_\nu} + \|d_{\bar{A}^{\varepsilon_\nu}} \bar{B}_t^\nu\|^2_{L^2(\Sigma)} \right) dt \\ &\leq \frac{c}{m_\nu} + \frac{c}{m_\nu} \int_{S^1} \left\|d_{A^{\varepsilon_\nu}} \left(\partial_t A^{\varepsilon_\nu} - d_{A^{\varepsilon_\nu}} \Psi^{\varepsilon_\nu}\right)\right\|^2_{L^2(\Sigma)} dt \\ &\leq \frac{c}{m_\nu} + \frac{c\varepsilon_\nu^{\frac{1}{2}}}{m_\nu}\end{aligned}$$

where the last step follows from the next claim. Then the L^4-norm of $\bar{B}_t^\nu$ converges to 0 by the last estimate in contradiction to (132).

Claim *For any perturbed Yang–Mills connection* $\Xi = A + \Psi dt$

$$\|d_A B_t\|_{L^2} \leq c\varepsilon^{\frac{1}{4}}$$

where we denote $\partial_t A - d_A \Psi$ *by* B_t.

Proof If we consider the identity

$$\begin{aligned}
&\varepsilon^2 \left\| \frac{1}{\varepsilon^2} d_A^* F_A - \nabla_t B_t - *X_t(A) \right\|_{L^2}^2 + \|\nabla_t F_A - d_A B_t\|_{L^2}^2 \\
&\quad = \frac{1}{\varepsilon^2} \left\| d_A^* F_A \right\|_{L^2}^2 + \varepsilon^2 \|\nabla_t B_t\|_{L^2}^2 + \varepsilon^2 \|X_t(A)\|_{L^2}^2 \\
&\qquad + \|\nabla_t F_A\|_{L^2}^2 + \|d_A B_t\|_{L^2}^2 - 2\varepsilon^2 \left\langle *X_t(A), \frac{1}{\varepsilon^2} d_A^* F_A - \nabla_t B_t \right\rangle \\
&\qquad - 2 \left\langle d_A^* F_A, \nabla_t B_t \right\rangle - 2 \left\langle \nabla_t F_A, d_A B_t \right\rangle,
\end{aligned} \tag{134}$$

we can remark that first line vanishes by the perturbed Yang–Mills equation (21) and by the Bianchi identity $\nabla_t F_A = d_A B_t$; in addition, the last line can be written as

$$-2 \left\langle d_A^* F_A, \nabla_t B_t \right\rangle - 2 \left\langle \nabla_t F_A, d_A B_t \right\rangle = 2 \left\langle F_A, [B_t \wedge B_t] \right\rangle$$

by the commutation formula (46). The identity (134) yields therefore to

$$\begin{aligned}
&\|d_A B_t\|_{L^2}^2 + \varepsilon^2 \|\nabla_t B_t\|_{L^2}^2 \\
&\quad \leq 2 \left| \langle d_A * X_t(A), F_A \rangle \right| + \varepsilon^2 \left| \langle *\nabla_t X_t(A), B_t \rangle \right| + c \|F_A\|_{L^2} \cdot \|B_t\|_{L^4}^2 \\
&\quad \leq c \|F_A\|_{L^2}^2 + \varepsilon^2 (1 + \|B_t\|_{L^2}^2) + c\varepsilon^{-\frac{1}{2}} \|F_A\|_{L^2} \cdot \|B_t\|_{1,2,\varepsilon}^2 \\
&\quad \leq c\varepsilon^2 (1 + \|B_t\|_{L^2}^2) + c\varepsilon^{\frac{1}{2}} \left(\|B_t\|_{L^2}^2 + \|d_A B_t\|_{L^2}^2 + \varepsilon^2 \|\nabla_t B_t\|_{L^2}^2 \right) \\
&\quad \leq c\varepsilon^{\frac{1}{2}} + c\varepsilon^{\frac{1}{2}} \left(\|d_A B_t\|_{L^2}^2 + \varepsilon^2 \|\nabla_t B_t\|_{L^2}^2 \right)
\end{aligned}$$

where we use the Hölder inequality and the Sobolev estimate in the second estimate and the assumption $\frac{1}{\varepsilon^2} \|F_A\|_{L^2}^2 + \|B_t\|_{L^2}^2 \leq 2b$ in the last two estimates. Thus choosing ε small enough the claim holds. □

Since we have found a contradiction for all the tree cases, we can conclude that

$$\sup_{t \in S^1} \|\partial_t A - d_A \Psi\|_{L^4}$$

is uniformly bounded for ε sufficiently small and thus the proof of the Theorem 14 is finished. □

11 Surjectivity of $\mathcal{T}^{b,\varepsilon}$

In the fifth chapter we defined the injective map $\mathcal{T}^{\varepsilon,b}$ in a unique way, in this one we show that it is also surjective provided that ε is chosen sufficiently small.

Theorem 16 *Let* $b > 0$ *be a regular value of* E^H. *Then there is a constant* $\varepsilon_0 > 0$ *such that*

$$\mathcal{T}^{\varepsilon,b} : \mathrm{Crit}_{E^H}^b \to \mathrm{Crit}_{\mathcal{YM}^{\varepsilon,H}}^b$$

is bijective for $0 < \varepsilon < \varepsilon_0$.

Proof The indirect proof will be divided in five steps. First, we assume that there is a decreasing sequence ε_ν, $\nu \to \infty$, converging to 0 and a sequence of perturbed Yang–Mills connections $\Xi^\nu = A^\nu + \Psi^\nu dt \in \mathrm{Crit}^b_{\mathcal{YM}^{\varepsilon_\nu,H}}$ that are not in the image of $\mathcal{T}^{\varepsilon_\nu,b}$. By the Theorems 12 and 14 the sequence satisfies

$$\|F_{A^\nu}\|_{L^\infty(\Sigma)} \leq c\varepsilon^{2-\frac{1}{p}}, \quad \left\|\partial_t A^\nu - d_{A^\nu}\Psi^\nu\right\|_{L^\infty(\Sigma)} \leq c, \tag{135}$$

$$\sup_{s\in S^1}\Big(\|F_{A^\nu}\|_{L^2(\Sigma)} + \left\|d^*_{A^\nu}F_{A^\nu}\right\|_{L^2(\Sigma)} + \left\|d_{A^\nu}d^*_{A^\nu}F_{A^\nu}\right\|_{L^2(\Sigma)}$$
$$+\varepsilon_\nu\left\|\nabla_t^{\Psi^\nu}F_{A^\nu}\right\|_{L^2(\Sigma)} + \varepsilon_\nu^2\left\|\nabla_t^{\Psi^\nu}\nabla_t^{\Psi^\nu}F_{A^\nu}\right\|_{L^2(\Sigma)}\Big) \leq c\varepsilon_\nu^{2-1/p}. \tag{136}$$

In the estimate (136) the constant c depends on $p \geq 2$ which can be taken as big as we want. In order to conclude the proof we will need to choose $p > 6$ as we will see in the proof of the fifth step. In step 1, for each Ξ^ν we will define a connection $\bar{\Xi}^\nu = \bar{A}^\nu + \bar{\Psi}^\nu dt$ near Ξ^ν, flat on the fibers, which satisfies, for a constant $c > 0$, $\left\|\pi_{\bar{A}^\nu}\left(\mathcal{F}^0\left(\bar{\Xi}^\nu\right)\right)\right\|_{L^p} \leq c\varepsilon_\nu^{1-1/p}$. Next, in the second step, we will find a representative Ξ^0 of a perturbed geodesic for which $\left\|\Xi^\nu - \Xi^0\right\|_{1,p,1} + \left\|\Xi^\nu - \Xi^0\right\|_{L^\infty} \leq c\varepsilon_\nu^{1-1/p}$ for a subsequence of $\{\Xi^\nu\}_{\nu\in\mathbb{N}}$ (step 3). Then, in step 5, we will improve this last estimate in order to apply the local uniqueness Theorem 9 which requires that the norms are bounded by $\delta\varepsilon$ for δ and ε sufficiently small; in this way we will have a contradiction, because a subsequence of $\{\Xi^\nu\}_{\nu\in\mathbb{N}}$ will turn out to be in the image of $\mathcal{T}^{\varepsilon_\nu,b}$.

Step 1. There are two positive constants c and ν_0 such that the following holds. For every Ξ^ν, $\nu > \nu_0$, there is a connection $\bar{\Xi}^\nu = \bar{A}^\nu + \bar{\Psi}^\nu dt$ which satisfies

$$i)\quad F_{\bar{A}^\nu} = 0, \qquad\qquad ii)\ d^*_{\bar{A}^\nu}(\partial_t\bar{A}^\nu - d_{\bar{A}^\nu}\bar{\Psi}^\nu) = 0,$$
$$iii)\ \left\|\Xi^\nu - \bar{\Xi}^\nu\right\|_{\bar{\Xi}^\nu,1,p,\varepsilon_\nu} \leq c\varepsilon_\nu^{2-\frac{1}{p}},\ iv)\ \left\|\pi_{\bar{A}^\nu}\left(\mathcal{F}^0\left(\bar{\Xi}^\nu\right)\right)\right\|_{L^p} \leq c\varepsilon_\nu^{1-1/p}.$$

Proof Since $\|F_{A^\nu}\|_{L^\infty(\Sigma)} \leq c\varepsilon^{2-\frac{1}{p}}$, by Lemma 12 there is a positive constant c such that for any A^ν there is a unique 0-form γ^ν which satisfies $F_{A^\nu+*d_{A^\nu}\gamma^\nu} = 0$, $\|d_{A^\nu}\gamma^\nu\|_{L^\infty(\Sigma)} \leq c\|F_{A^\nu}\|_{L^\infty(\Sigma)}$. We denote $\bar{\Xi}^\nu := \bar{A}^\nu + \bar{\Psi}^\nu dt$ where $\bar{A}^\nu := A^\nu + *d_{A^\nu}\gamma^\nu$, $\alpha^\nu := *d_{A^\nu}\gamma^\nu$ and $\bar{\Psi}^\nu := \Psi^\nu + \psi^\nu$ is the unique 0-form such that $d^*_{\bar{A}^\nu}(\partial_t\bar{A}^\nu - d_{\bar{A}^\nu}\bar{\Psi}^\nu) = 0$; $\bar{\Psi}^\nu$ is well defined because $d^*_A d_A : \Omega^0(\Sigma, \mathfrak{g}_P) \to \Omega^0(\Sigma, \mathfrak{g}_p)$ is bijective for any flat connection A. Hence, α^ν satisfies the estimate

$$\|\alpha^\nu\|_{L^\infty(\Sigma)} = \|d_{A^\nu}\gamma^\nu\|_{L^\infty(\Sigma)} \leq c\|F_{A^\nu}\|_{L^\infty(\Sigma)} \leq c\varepsilon_\nu^{2-\frac{1}{p}}. \tag{137}$$

Since the Ξ^ν is a perturbed Yang–Mills connection, i.e.

$$\frac{1}{\varepsilon_\nu^2}d^*_{A^\nu}F_{A^\nu} = \nabla_t^{\Psi^\nu}(\partial_t A^\nu - d_{A^\nu}\Psi^\nu) + *X(A^\nu), \tag{138}$$

we have that the connections $\bar{\Xi}^\nu$ satisfy

$$\begin{aligned}&\nabla_t^{\bar{\Psi}^\nu}(\partial_t\bar{A}^\nu - d_{\bar{A}^\nu}\bar{\Psi}^\nu) + *X(\bar{A}^\nu)\\ &\quad= \nabla_t^{\Psi^\nu}(\partial_t A^\nu - d_{A^\nu}\Psi^\nu) + *X(A^\nu)\\ &\qquad+[\psi^\nu, (\partial_t A^\nu - d_{A^\nu}\Psi^\nu)] + \nabla_t^{\bar{\Psi}^\nu}(\nabla_t^{\bar{\Psi}^\nu}\alpha^\nu - d_{A^\nu}\psi^\nu)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon_\nu^2} d_{\bar{A}^\nu}^* F_{A^\nu} - \frac{1}{\varepsilon_\nu^2} * \left[\alpha^\nu \wedge * F_{A^\nu}\right] + 2[\psi^\nu, (\partial_t A^\nu - d_{A^\nu}\Psi^\nu)] \\
&\quad + \nabla_t^{\bar{\Psi}^\nu} \nabla_t^{\bar{\Psi}^\nu} \alpha^\nu - d_{A^\nu} \nabla_t^{\Psi^\nu} \psi^\nu - [\psi^\nu, d_{A^\nu}\psi^\nu]
\end{aligned}
$$

where in the last equality we used (138) and the commutation formula (46); thus,

$$
\begin{aligned}
\pi_{\bar{A}^\nu}\left(\mathcal{F}^0(\bar{A}^\nu, \bar{\Psi}^\nu)\right) &= -\pi_{\bar{A}^\nu}\left(\nabla_t^{\bar{\Psi}^\nu}(\partial_t \bar{A}^\nu - d_{\bar{A}^\nu}\bar{\Psi}^\nu) + *X(\bar{A}^\nu)\right) \\
&= \pi_{\bar{A}^\nu}\left(* \frac{1}{\varepsilon_\nu^2}\left[\alpha^\nu \wedge * F_{A^\nu}\right] - 2[\psi^\nu, (\partial_t A^\nu - d_{A^\nu}\Psi^\nu)]\right) \\
&\quad - \pi_{\bar{A}^\nu}\left([\psi^\nu, d_{A^\nu}\psi^\nu] + \nabla_t^{\bar{\Psi}^\nu}\nabla_t^{\bar{\Psi}^\nu}\alpha^\nu + [\alpha^\nu, \nabla_t^{\Psi^\nu}\phi^\nu]\right).
\end{aligned}
$$

Therefore, by (137) and the next lemma,

$$
\begin{aligned}
&\left\|\pi_{\bar{A}^\nu}\left(\mathcal{F}^0(\bar{A}^\nu, \bar{\Psi}^\nu)\right)\right\|_{L^p} \\
&\leq \left\|\frac{1}{\varepsilon_\nu^2}\pi_{\bar{A}^\nu}\left(*\left[\alpha^\nu \wedge * F_{A^\nu}\right]\right)\right\|_{L^p} + \left\|\pi_{\bar{A}^\nu}\left(2[\psi^\nu, (\partial_t A^\nu - d_{A^\nu}\Psi^\nu)]\right)\right\|_{L^p} \\
&+ \left\|\pi_{\bar{A}^\nu}\left(\nabla_t^{\bar{\Psi}^\nu}\nabla_t^{\bar{\Psi}^\nu}\alpha^\nu + [\alpha^\nu, \nabla_t^{\Psi^\nu}\phi^\nu] - [\psi^\nu, d_{A^\nu}\psi^\nu]\right)\right\|_{L^p} \\
&\leq c\varepsilon_\nu^{2-\frac{2}{p}} + \left\|\pi_{\bar{A}^\nu}\left(\nabla_t^{\Psi^\nu}\nabla_t^{\bar{\Psi}^\nu}\alpha^\nu\right)\right\|_{L^p}
\end{aligned}
$$

where

$$
\begin{aligned}
&\left\|\pi_{\bar{A}^\nu}\left(\nabla_t^{\Psi^\nu}\nabla_t^{\bar{\Psi}^\nu}\alpha^\nu\right)\right\|_{L^p} \\
&\leq \left\|\pi_{\bar{A}^\nu}\left(\nabla_t^{\Psi^\nu}[\psi^\nu, \alpha^\nu] + *\nabla_t^{\Psi^\nu}[(\partial_t A^\nu - d_{A^\nu}\Psi^\nu), \gamma^\nu]\right)\right\|_{L^p} \\
&\quad + \left\|\pi_{\bar{A}^\nu}\left(d_{A^\nu}\nabla_t^{\Psi^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu + [(\partial_t A^\nu - d_{A^\nu}\Psi^\nu), \nabla_t^{\Psi^\nu}\gamma^\nu]\right)\right\|_{L^p} \\
&\leq c\varepsilon_\nu^{1-1/p}
\end{aligned}
$$

follows from Lemma 9 and hence

$$
\left\|\pi_{\bar{A}^\nu}\left(\mathcal{F}^0(A^\nu, \Psi^\nu)\right)\right\|_{0,p,\varepsilon} \leq c\varepsilon_\nu^{1-1/p}. \tag{139}
$$

The estimate $\left\|\Xi^\nu - \bar{\Xi}^\nu\right\|_{\bar{\Xi}^\nu,1,p,\varepsilon_\nu} \leq c\varepsilon_\nu^{2-\frac{1}{p}}$ follows from the Lemma 9. This completes the proof of the first step. □

Lemma 9 *There are constants* $c > 0$, $\bar{\varepsilon}_0 > 0$ *such that*

$$
\begin{aligned}
\left\|\psi^\nu\right\|_{L^\infty(\Sigma)} + \left\|d_{A^\nu}\psi^\nu\right\|_{L^p(\Sigma)} &\leq c\varepsilon_\nu^{2-1/p}, \\
\left\|\nabla_t^{\Psi^\nu}\alpha^\nu\right\|_{L^p(\Sigma)} + \left\|\nabla_t^{\Psi^\nu}\gamma^\nu\right\|_{L^\infty(\Sigma)} &\leq c\varepsilon_\nu^{1-1/p}, \\
\left\|\nabla_t^{\Psi^\nu}\psi^\nu\right\|_{L^\infty(\Sigma)} + \varepsilon\left\|\nabla_t^{\Psi^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\right\|_{L^\infty(\Sigma)} &\leq c\varepsilon_\nu^{1-1/p}
\end{aligned}
$$

for any $0 < \varepsilon_\nu < \bar{\varepsilon}_0$.

Proof Since the Yang–Mills connections Ξ^ν satisfy

$$
d_{A^\nu}^*\left(\partial_t A^\nu - d_{A^\nu}\Psi^\nu\right) = 0,
$$

from the definition of ψ^ν we have

$$\begin{aligned}0 &= d^*_{\bar{A}^\nu}\left(\partial_t \bar{A}^\nu - d_{\bar{A}^\nu}\bar{\Psi}^\nu\right)\\ &= -*\left[\alpha^\nu \wedge *\left(\partial_t A^\nu - d_{A^\nu}\Psi^\nu\right)\right] + d^*_{\bar{A}^\nu}\nabla_t^{\Psi^\nu}\alpha^\nu - d^*_{\bar{A}^\nu} d_{\bar{A}^\nu}\psi^\nu \end{aligned} \tag{140}$$

where

$$\begin{aligned} d^*_{\bar{A}^\nu}\nabla_t^{\Psi^\nu}\alpha^\nu &= -*[(\partial_t A^\nu - d_{A^\nu}\Psi^\nu)\wedge *\alpha^\nu] - *[\alpha^\nu \wedge *\nabla_t^{\Psi^\nu}\alpha^\nu] + \nabla_t^{\Psi^\nu} d^*_{A^\nu}\alpha^\nu,\\ \nabla_t^{\Psi^\nu} d^*_{A^\nu}\alpha^\nu &= \nabla_t^{\Psi^\nu} d^*_{A^\nu} d^*_{A^\nu} * \gamma^\nu = \nabla_t^{\Psi^\nu} * [F_{A^\nu}\wedge \gamma^\nu]\\ &= *[\nabla_t^{\Psi^\nu} F_{A^\nu}\wedge\gamma^\nu] + *[F_{A^\nu}\wedge \nabla_t^{\Psi^\nu}\gamma^\nu]. \end{aligned}$$

Since we know that $\|\alpha^\nu\|_{L^\infty(\Sigma)} + \|\gamma^\nu\|_{L^\infty(\Sigma)} \le c\varepsilon_\nu^{2-\frac{1}{p}}$, the proof of the first two inequalities of the lemma is completed by showing that there exists a constant c such that

$$\|\nabla_t^{\Psi^\nu}\alpha^\nu\|_{L^p(\Sigma)} + \|\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^p(\Sigma)} \le c\varepsilon_\nu^{1-1/p} \tag{141}$$

and estimating the norms of ψ^ν and of $d_{\bar{A}^\nu}\psi^\nu$ using (140):

$$\begin{aligned}\|\psi^\nu\|_{L^\infty(\Sigma)} &\le c\|d_{\bar{A}^\nu}\psi^\nu\|_{L^p(\Sigma)} \le c\|d^*_{\bar{A}^\nu} d_{\bar{A}^\nu}\psi^\nu\|_{L^2(\Sigma)}\\ &= \|-*\left[\alpha^\nu \wedge *\left(\partial_t A^\nu - d_{A^\nu}\Psi^\nu\right)\right] + d^*_{\bar{A}^\nu}\nabla_t^{\Psi^\nu}\alpha^\nu\|_{L^2(\Sigma)}\\ &\le c\|\alpha^\nu\|_{L^2(\Sigma)} + \|\alpha^\nu\|_{L^\infty(\Sigma)}\|\nabla_t^{\Psi^\nu}\alpha^\nu\|_{L^2(\Sigma)}\\ &\quad + \|\gamma^\nu\|_{L^\infty(\Sigma)}\|\nabla_t^{\Psi^\nu}F_{A^\nu}\|_{L^2(\Sigma)} + \|F_{A^\nu}\|_{L^\infty(\Sigma)}\|\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^2(\Sigma)}\\ &\le c\varepsilon_\nu^{2-\frac{1}{p}}. \end{aligned}$$

In order to show (141) we derive

$$F_{A^\nu} + d_{A^\nu} * d_{A^\nu}\gamma^\nu + \frac{1}{2}[d_{A^\nu}\gamma^\nu \wedge d_{A^\nu}\gamma^\nu] = 0$$

by $\nabla_t^{\Psi^\nu}$ and we obtain

$$\begin{aligned} d_{A^\nu} * d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu = &-\nabla_t^{\Psi^\nu}F_{A^\nu} - [d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu \wedge d_{A^\nu}\gamma^\nu]\\ &- [[(\partial_t A^\nu - d_{A^\nu}\Psi^\nu)\wedge\gamma^\nu]\wedge d_{A^\nu}\gamma^\nu]\\ &- [(\partial_t A^\nu - d_{A^\nu}\Psi^\nu)\wedge * d_{A^\nu}\gamma^\nu] \end{aligned} \tag{142}$$

and hence, by (135),

$$\begin{aligned} &\|d_{A^\nu} * d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^2(\Sigma)}\\ &\le c\|\nabla_t^{\Psi^\nu}F_{A^\nu}\|_{L^2(\Sigma)} + c\|\alpha^\nu\|_{L^\infty(\Sigma)}\|d_{A^\nu} * d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^2(\Sigma)}\\ &\quad + c\left\|\partial_t A^\nu - d_{A^\nu}\Psi^\nu\right\|_{L^\infty(\Sigma)}\|\alpha^\nu\|_{L^\infty(\Sigma)}\left(1 + \|\gamma^\nu\|_{L^2(\Sigma)}\right)\\ &\le c\left(\|\nabla_t^{\Psi^\nu}F_{A^\nu}\|_{L^2(\Sigma)} + \|\alpha^\nu\|_{L^\infty(\Sigma)}\right) + c\varepsilon^{\frac{2}{-}\frac{1}{p}}\|d_{A^\nu} * d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^2(\Sigma)}. \end{aligned}$$

Choosing ε sufficiently small, we have by (136),

$$\|d_{A^\nu} * d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^2(\Sigma)} \le c\varepsilon_\nu^{1-\frac{1}{p}}$$

which yields to

$$\begin{aligned}\|\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^\infty(\Sigma)} &\le c\|d_{A^\nu} * d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^2(\Sigma)}\\ &\le c\left(\|\nabla_t^{\Psi^\nu}F_{A^\nu}\|_{L^2(\Sigma)} + \|\alpha^\nu\|_{L^\infty(\Sigma)}\right) \le c\varepsilon_\nu^{1-1/p}\end{aligned}$$

by Lemma 11 and

$$\begin{aligned}\|\nabla_t^{\Psi^\nu}\alpha^\nu\|_{L^p(\Sigma)} &= \|\nabla_t^{\Psi^\nu} d_{A^\nu}\gamma^\nu\|_{L^p(\Sigma)}\\ &\le \|d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^p(\Sigma)} + \|[(\partial_t A^\nu - d_{A^\nu}\Psi^\nu), \gamma^\nu]\|_{L^p(\Sigma)}\\ &\le c\|d_A * d_{A^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\|_{L^2(\Sigma)} + c\|\gamma^\nu\|_{L^p(\Sigma)} \le c_2\varepsilon_\nu^{1-1/p}.\end{aligned}$$

Analogously, one can obtain the last inequality of the lemma; the starting point is to derive (140) and (142) by $\nabla_t^{\Psi^\nu}$ and to use the estimate

$$\left\|\nabla_t^{\Psi^\nu}\nabla_t^{\Psi^\nu} F_{A^\nu}\right\|_{L^2(\Sigma)} \le c_2\varepsilon^{-1/p}$$

in order to show

$$\left\|\nabla_t^{\Psi^\nu}\psi^\nu\right\|_{L^\infty(\Sigma)} + \varepsilon_\nu\left\|\nabla_t^{\Psi^\nu}\nabla_t^{\Psi^\nu}\gamma^\nu\right\|_{L^\infty(\Sigma)} \le c\varepsilon_\nu^{1-1/p}.$$

□

In the following, by the Nash embedding theorem, we consider $\mathcal{M}^g(P)$ to be a compact submanifold of $\mathbb{R}^n$.

Step 2. The sequence $\left\{u^\nu := \left[\bar{A}^\nu\right]\right\}_{\nu\in\mathbb{N}}$ has a subsequence, still denoted by u^ν, which converges to a perturbed geodesic u^0 respect to the norm $\|\cdot\|_{W^{1,p}}$ or more precisely

$$\left\|u^\nu - u^0\right\|_{W^{1,p}} \le c\varepsilon_\nu^{1-1/p}$$

for a constant $c > 0$.

Proof Since $F_{\bar{A}^\nu} = 0$, the vector $\partial_t\bar{A}^\nu$ lies on the tangent space $T_{\bar{A}^\nu}\mathcal{A}_0(P)$ and hence in the kernel of $d_{\bar{A}^\nu}$; thus $d_{\bar{A}^\nu}(\partial_t\bar{A}^\nu - d_{\bar{A}^\nu}\bar{\Psi}^\nu) = 0$. Every $[\bar{A}^\nu]$ is therefore a curve in the moduli space $\mathcal{M}^g(P)$ with velocity $\partial_t\bar{A}^\nu - d_{\bar{A}^\nu}\bar{\Psi}^\nu$; moreover it approximates a geodesic in the sense of inequality (139). Therefore $\{u^\nu\}_{\nu\in\mathbb{N}}$ is a bounded Palais–Smale sequence and hence, using next lemma, it has a strong convergent subsequence that converge in the norm $\|\cdot\|_{W^{1,p}}$ to a perturbed geodesic u^0 and $\|u^\nu - u^0\|_{W^{1,p}} \le c\varepsilon_\nu^{1-\frac{1}{p}}$. □

Lemma 10 *Let $p \ge 2$ and $\mathcal{M}$ be a compact embedded manifold. We choose the energy*

$$E(u) = \frac{1}{2}\int_0^1 \left(|\nabla u|^2 + H_t(u)\right) dt$$

for any $u \in W^{1,p}(S^1, \mathcal{M})$ where $H_t : \mathcal{M} \to \mathbb{R}$ is a smooth Hamiltonian. For every bounded sequence $\{u^\nu\}_{\nu\in\mathbb{N}} \subset W^{1,p}(S^1, \mathcal{M})$ which satisfies

$$\|dE(u^\nu)\|_{L^p} \to 0$$

there is a critical curve $u_\infty \in W^{1,p}(S^1, \mathcal{M})$ such that for a subsequence $\{u^{\iota_\nu}\}_{\nu\in\mathbb{N}} \subset \{u^\nu\}_{\nu\in\mathbb{N}}$ we have

1. $\|u^{\iota_\nu} - u^0\|_{W^{1,p}} \to 0 \quad (k \to \infty)$;
2. *The $\{LE(u^{\iota_\nu})\}_{\nu\in\mathbb{N}}$, where LE denote the linearisation of dE, converges in L^p to the Jacobi operator of u^0;*
3. *If the Jacobi operator of u_∞ is invertible, then there is a constant $c > 0$ such that $\|u^{\iota_\nu} - u^0\|_{W^{1,p}} \le c\|dE(u^\nu)\|_{L^p}$.*

Proof 1. The energy functional E satisfies the Palais–Smale condition for the norm $\|\cdot\|_{W^{1,2}}$: We refer the reader to [14], theorem 4.4, for the proof in the case $\mathcal{M}$ is a surface and $H_t = 0$, but the proof applied also for the general case. Therefore, $\{u^\nu\}_{\nu\in\mathbb{N}}$ has a subsequence, still denoted by $\{u^\nu\}_{\nu\in\mathbb{N}}$, which converges to a perturbed geodesic u^0 in $W^{1,2}(S^1,\mathcal{M})$. It remains to prove that the sequence converges to u^0 also in $\|\cdot\|_{W^{1,p}}$, in fact

$$\begin{aligned}\|u^\nu - u^0\|_{W^{1,p}} &\le \sup_{v\in W^{1,q},\|v\|_{W^{1,q}}=1} \int_0^1 \Big(\langle\nabla(u^\nu - u^0), \nabla v\rangle + \langle u^\nu - u^0, v\rangle\Big) dt\\ &= \sup_{v\in W^{1,q},\|v\|_{W^{1,q}}=1} \Bigg(-\int_0^1 \langle dH(u^\nu) - dH_t(u^0), v\rangle dt\\ &\quad + \int_0^1 \langle \Delta(u^\nu) + dH_t(u^\nu), v\rangle dt + \int_0^1 \langle u^\nu - u^0, v\rangle dt \Bigg)\end{aligned}$$

converges to 0.

2. The L^p convergence of $dE(u^\nu)$ implies the convergence of $\nabla_t \dot{u}^\nu$ because

$$||\nabla_t\dot{u}^\nu - \nabla_t\dot{u}^0||_{L^p} \le ||dE(u^\nu) - dE(u^0)||_{L^p} + ||dH(u^\nu) - dH(u^0)||_{L^p}$$

goes to 0 for $\nu \to \infty$. We denote by R the Riemann tensor of the manifold $\mathcal{M}$ and by Π the projection on its tangent space. Then the linearisation of dE respect to the loops u^ν is (cf. appendix B in [15])

$$LE(u^\nu)X(u^\nu) = -\nabla_{\dot{u}^\nu}\nabla_{\dot{u}^\nu}X(u^\nu) - R(X(u^\nu), \dot{u}^\nu)\dot{u}^\nu - \nabla_{X(u^\nu)}\nabla H_t(u^\nu)$$

for any vector field X on $\mathcal{M}$ and the first term can be written as

$$\begin{aligned}\nabla_{\dot{u}^\nu}\nabla_{\dot{u}^\nu}X(u^\nu) &= \nabla_{\dot{u}^\nu}\left(\Pi(u^\nu)dX(u^\nu)\dot{u}^\nu\right)\\ &= \Pi(u^\nu)\left(d\Pi(u^\nu)\dot{u}^\nu\right)\left(dX(u^\nu)\dot{u}^\nu\right)\\ &\quad +\Pi(u^\nu)d^2X(u^\nu)\dot{u}^\nu\dot{u}^\nu)\\ &\quad +\Pi(u^\nu)dX(u^\nu)\nabla_{\dot{u}^\nu}\dot{u}^\nu.\end{aligned}$$

Thus, for a constant $c > 0$,

$$\begin{aligned}\left\|LE(u^\nu) - LE(u^0)\right\|_{L^p} &\le c\left(\left\|u^\nu - u^0\right\|_{L^p} + \left\|\dot{u}^\nu - \dot{u}^0\right\|_{L^p}\right)\\ &\quad +c\left\|\nabla_{\dot{u}^\nu}\dot{u}^\nu - \nabla_{\dot{u}^0}\dot{u}^0\right\|_{L^p}.\end{aligned}$$

The sequence $\{LE(u^\nu)\}_{k\in\mathbb{N}}$ converges therefore to the Jacobi operator of u^0 in L^p.

3. The third conclusion of the theorem can be proved using the following theorem (see proposition A.3.4. in [10]). In our case we chose

$$\begin{aligned} f : W^{2,p}((u^\nu)^*T\mathcal{M}) &\to L^2((u^\nu)^*T\mathcal{M})\\ x &\mapsto f(x) := g_x(\mathcal{F}_0(\exp_{u^\nu}(x)))\end{aligned}$$

where $g_x : L^p(\exp_{u^\nu}(x)^*T\mathcal{M}) \to L^p((u^\nu)^*T\mathcal{M})$ is the parallel transport along $t \mapsto \exp_{u^\nu}((1-t)x)$ and the proof is complete.

□

Theorem 17 *Let X and Y be Banach spaces, $U \subset X$ be an open set, and $f : U \to Y$ be a continuously differentiable map. Let $x_0 \in U$ be such that $D := df(x_0) : X \to Y$ is* *surjective and has a (bounded linear) right inverse $Q : Y \to X$. Choose positive constants δ and c such that $\|Q\| \le c$, $B_\delta(x_0; X) \subset U$, and*

$$\|x - x_0\| < \delta \quad \Rightarrow \quad \|df(x) - D\| \le \frac{1}{2c}.$$

Suppose that $x_1 \in X$ satisfies

$$\|f(x_1)\| < \frac{\delta}{4c}, \quad \|x_1 - x_0\| < \frac{\delta}{8}.$$

Then there exists a unique $x \in X$ such that

$$f(x) = 0, \quad x - x_1 \in im\, Q, \quad \|x - x_0\| \le \delta.$$

Moreover, $\|x - x_1\| \le 2c\|f(x_1)\|$.

Step 3. There is a lift Ξ^0 of the closed geodesic u^0 and a sequence $g_\nu \subset \mathcal{G}_0^{2,p}(P \times S^1)$ such that

$$\begin{aligned} &\left\|g_\nu^*\Xi^\nu - \Xi^0\right\|_{1,p,1} + \left\|g_\nu^*\Xi^\nu - \Xi^0\right\|_{L^\infty} \le c\varepsilon_\nu^{1-1/p}, \\ &\|d_{A^0}(g_\nu^*A^\nu - A^0)\|_{L^p} \le c\varepsilon_\nu^{2-2/p}. \end{aligned} \tag{143}$$

and $d_{A^0}^*(g_\nu^*A^\nu - A^0)\|_{L^2} = 0$. For expositional reasons we will still denote by Ξ^ν the sequence $g_\nu^*\Xi^\nu$.

Proof We choose now a representative $\Xi^0 = A^0 + \Psi^0 dt$ of the geodesic u^0. Since the sequence of curves on the moduli space converges to a geodesic $[\Xi^0]$ in $W^{1,p}$, i.e.

$$\left\|\left[\bar{\Xi}^\nu\right] - [\Xi^0]\right\|_{W^{1,p}(S^1,\mathcal{M})} \le c\varepsilon_\nu^{1-1/p}, \tag{144}$$

by the Sobolev embedding theorem we have that

$$\left\|\left[\bar{\Xi}^\nu\right] - [\Xi^0]\right\|_{L^\infty} \le c\varepsilon_\nu^{1-1/p}.$$

Therefore there is a sequence $g_\nu \subset \mathcal{G}_0^{2,p}(P \times S^1)$ such that

$$d_{A^0}^*\left(g_\nu^*A^\nu - A^0\right) = 0 \tag{145}$$

and in order to symplify the exposition we still denote the sequence $g_\nu^*\Xi^\nu$ by Ξ^ν. The condition (145) means that we choose the closest connection in the orbit of A_ν to A_0 respect to the $L^2(\Sigma)$-norm. The existence of g_ν is assured by the Lemma 1 and by the local slice theorem (see theorem 8.1 in [17]). Therefore $\left\|\bar{\Xi}^\nu - \Xi^0\right\|_{L^\infty} \le c\varepsilon_\nu^{1-1/p}$ and thus by the first step

$$\left\|\Xi^\nu - \Xi^0\right\|_{L^\infty} \le c\varepsilon_\nu^{1-1/p}. \tag{146}$$

Since

$$d_{A^0}\left(A^\nu - A^0\right) = F_{A^\nu} - \frac{1}{2}\left[\left(A^\nu - A^0\right) \wedge \left(A^\nu - A^0\right)\right],$$

we have the estimate

$$\left\|d_{A^0}\left(A^\nu - A^0\right)\right\|_{L^p} \le \|F_{A^\nu}\|_{L^p} + c\|A^\nu - A^0\|_{L^\infty}\|A^\nu - A^0\|_{L^p} \le c\varepsilon_\nu^{2-2/p}. \tag{147}$$

Next, we remark, using $\nabla_t := \partial_t + [\Psi^0, \cdot]$, that

$$0 = \nabla_t d_{A^0}^*\left(A^\nu - A^0\right) = d_{A^0}^*\nabla_t\left(A^\nu - A^0\right) + *\left[\left(\partial_t A^0 - d_{A^0}\Psi^0\right) \wedge *\left(A^\nu - A^0\right)\right],$$

thus,

$$\begin{aligned}
d_{A^0}^* d_{A^0}\left(\Psi^\nu-\Psi^0\right) &= d_{A^0}^*\left(\partial_t A^0-d_{A^0}\Psi^0\right)-d_{A^\nu}^*\left(\partial_t A^\nu-d_{A^\nu}\Psi^\nu\right)\\
&\quad+d_{A^0}^*\nabla_t\left(A^\nu-A^0\right)-d_{A^0}^*\left[\left(A^\nu-A^0\right)\wedge\left(\Psi^\nu-\Psi^0\right)\right]\\
&\quad-*\left[(A^\nu-A^0)\wedge*\left(\partial_t A^\nu-d_{A^\nu}\Psi^\nu\right)\right]\\
&=-*\left[\left(A^\nu-A^0\right)\wedge*\left(\partial_t A^\nu-d_{A^\nu}\Psi^\nu\right)\right]\\
&\quad-*\left[\left(A^\nu-A^0\right)\wedge*\left(\partial_t A^0-d_{A^0}\Psi^0\right)\right]\\
&\quad-*\left[*(A^\nu-A^0)\wedge d_{A^0}(\Psi^\nu-\Psi^0)\right]
\end{aligned}\tag{148}$$

allows us to compute the estimate using (135)

$$\begin{aligned}
&\left\|d_{A^0}\left(\Psi^\nu-\Psi^0\right)\right\|_{L^p}+\left\|\Psi^\nu-\Psi^0\right\|_{L^p}\leq c\left\|d_{A^0}*d_{A^0}\left(\Psi^\nu-\Psi^0\right)\right\|_{L^p}\\
&\quad\leq c\|A^\nu-A^0\|_{L^p}+\|A^\nu-A^0\|_{L^\infty}\|d_{A^0}(A^\nu-A^0)\|_{L^p}\leq c\varepsilon_\nu^{1-1/p}.
\end{aligned}\tag{149}$$

Furthermore, since, by (144),

$$\begin{aligned}
\left\|\left(\partial_t A^\nu-d_{A^\nu}\Psi^\nu\right)-\left(\partial_t A^0-d_{A^0}\Psi^0\right)\right\|_{L^p}&\leq c\varepsilon_\nu^{1-1/p},\\
\left\|\nabla_t\left(A^\nu-A^0\right)\right\|_{L^p}&\leq c\varepsilon_\nu^{1-1/p}.
\end{aligned}\tag{150}$$

On the other side, we have

$$\begin{aligned}
d_{A^0}^* d_{A^0}\nabla_t\left(\Psi^\nu-\Psi^0\right) &= \nabla_t d_{A^0}^* d_{A^0}\left(\Psi^\nu-\Psi^0\right)\\
&\quad-*\left[\left(\partial_t A^0-d_{A^0}\Psi^0\right)\wedge*d_{A^0}\left(\Psi^\nu-\Psi^0\right)\right]\\
&\quad+*\left[d_{A^0}*\left(\partial_t A^0-d_{A^0}\Psi^0\right)\wedge\left(\Psi^\nu-\Psi^0\right)\right]\\
&\quad-*\left[*\left(\partial_t A^0-d_{A^0}\Psi^0\right)\wedge d_{A^0}\left(\Psi^\nu-\Psi^0\right)\right]
\end{aligned}$$

and deriving (148) by ∇_t we obtain

$$\begin{aligned}
\left\|\nabla_t\left(\Psi^\nu-\Psi^0\right)\right\|_{L^p} &\leq \left\|d_{A^0}^* d_{A^0}\nabla_t\left(\Psi^\nu-\Psi^0\right)\right\|_{L^p}\\
&\leq c\|d_{A^0}\nabla_t(\Psi^\nu-\Psi^0)\|_{L^p}+c\left\|\nabla_t\left(A^\nu-A^0\right)\right\|_{L^p}\\
&\quad+\left\|\nabla_t\left(A^\nu-A^0\right)\right\|_{L^{2p}}\left\|d_{A^0}\left(\Psi^\nu-\Psi^0\right)\right\|_{L^{2p}}\\
&\quad+c\|A^\nu-A^0\|_{L^\infty}\left(1+\frac{1}{\varepsilon^2}\left\|d_{A^n u}d_{A^\nu}^* F_{A^\nu}\right\|_{L^2(\Sigma)}\right)\\
&\quad+\|A^\nu-A^0\|_{L^\infty}\left\|d_{A^0}^* d_{A^0}\nabla_t\left(\Psi^\nu-\Psi^0\right)\right\|_{L^p}
\end{aligned}\tag{151}$$

where in the second estimate we use that, by the perturbed Yang–Mills equations,

$$\begin{aligned}
\|\nabla_t^{\Psi^\nu}(\partial_t A^\nu-d_{A^\nu}\Psi^\nu)\|_{L^p} &\leq c+c\frac{1}{\varepsilon^2}\left\|d_{A^\nu}^* F_{A^\nu}\right\|_{L^p(\Sigma)}\\
&\leq c+c\frac{1}{\varepsilon^2}\left\|d_{A^n u}d_{A^\nu}^* F_{A^\nu}\right\|_{L^2(\Sigma)};
\end{aligned}$$

thus,

$$\left\|\nabla_t\left(\Psi^\nu-\Psi^0\right)\right\|_{L^p}\leq c\varepsilon_\nu^{1-1/p}.\tag{152}$$

Finally by the estimates (146), (147), (145), (149), (150) and (152) we have

$$\left\|\Xi^\nu-\Xi^0\right\|_{1,p,1}+\varepsilon_\nu^{1/p}\left\|\Xi^\nu-\Xi^0\right\|_{L^\infty}\leq c\varepsilon_\nu^{1-1/p}\tag{153}$$

wich proves the third step. □

Step 4. Let $p > 3$. There is sequence $\{g_\nu\}_{\nu\in\mathbb{N}}$ of gauge transformations $g_\nu \in \mathcal{G}_0^{2,p}(P\times S^1)$ such that

$$d^{*_{\varepsilon_\nu}}_{\Xi^0}(g_\nu^*\Xi^\nu - \Xi^0) = 0, \tag{154}$$

$$\left\|d^*_{A^0}(g_\nu^* A^\nu - A^0)\right\|_{L^p} \le c\varepsilon_\nu^{3-1/p}, \quad \left\|d_{A^0}(g_\nu^* A^\nu - A^0)\right\|_{L^p} \le c\varepsilon_\nu^{2-2/p}, \tag{155}$$

and

$$\left\|g_\nu^*\Xi^\nu - \Xi^0\right\|_{1,p,1} + \varepsilon_\nu^{1/p}\left\|g_\nu^*\Xi^\nu - \Xi^0\right\|_{L^\infty} \le c\varepsilon_\nu^{1-1/p}. \tag{156}$$

Proof By the last step the perturbed Yang–Mills connections Ξ^ν, that satisfy the estimate (153) and in addition

$$\varepsilon_\nu^2\left\|d^{*_\varepsilon}_{\Xi^0}(\Xi^\nu - \Xi^0)\right\|_{L^p} \le \left\|d^*_{A^0}(A^\nu - A^0)\right\|_{L^p} + \varepsilon_\nu^2\left\|\nabla_t(\Psi^\nu - \Psi^0)\right\|_{L^p} \le c\varepsilon_\nu^{3-1/p},$$

$$\left\|\Xi^\nu - \Xi^0\right\|_{0,p,\varepsilon} \le \left\|\Xi^\nu - \Xi^0\right\|_{1,p,\varepsilon} \le c\varepsilon_\nu^{1-1/p} \le \delta_0\varepsilon_\nu^{1/p}$$

for all $0 < \varepsilon_\nu \le \varepsilon_0$, $c\varepsilon_0^{1-2/p} \le \delta_0$ and the δ_0 given in Theorem 11; hence Ξ^ν, Ξ^0 satisfy the assumption (88) of Theorem 11 with $q = p$. Therefore by this last theorem we can find a sequence $g_\nu \in \mathcal{G}_0^{2,p}(P\times S^1)$ such that

$$d^{*_{\varepsilon_\nu}}_{\Xi^0}(g_\nu^*\Xi^\nu - \Xi^0) = 0$$

and

$$\left\|g_\nu^*\Xi^\nu - \Xi^\nu\right\|_{1,p,\varepsilon} \le c\varepsilon^2\left\|d^{*_\varepsilon}_{\Xi^0}(\Xi^\nu - \Xi^0)\right\|_{L^p} \le c\varepsilon_\nu^{3-1/p} \tag{157}$$

and therefore, by the Sobolev Theorem 4,

$$\begin{aligned}\varepsilon_\nu^{1/p}\left\|g_\nu^*\Xi^\nu - \Xi^0\right\|_{\infty,\varepsilon} &\le c\left\|g_\nu^*\Xi^\nu - \Xi^0\right\|_{1,p,\varepsilon}\\ &\le c\left(\left\|g_\nu^*\Xi^\nu - \Xi^\nu\right\|_{1,p,\varepsilon} + \left\|\Xi^\nu - \Xi^0\right\|_{1,p,\varepsilon}\right)\\ &\le c\varepsilon_\nu^{1-1/p}.\end{aligned}$$

The estimates (153), (157) and the triangular inequality yield also to

$$\begin{aligned}\left\|d^*_{A^0}(g_\nu^*A^\nu - A^0)\right\|_{L^p} &\le \left\|d^*_{A^0}(A^\nu - A^0)\right\|_{L^p} + \left\|d^*_{A^0}(g_\nu^*A^\nu - A^\nu)\right\|_{L^p} \le c\varepsilon_\nu^{3-1/p},\\ \left\|d_{A^0}(g_\nu^*A^\nu - A^0)\right\|_{L^p} &\le \left\|d_{A^0}(A^\nu - A^0)\right\|_{L^p} + \left\|d_{A^0}(g_\nu^*A^\nu - A^\nu)\right\|_{L^p} \le c\varepsilon_\nu^{2-2/p},\\ \left\|g_\nu^*\Xi^\nu - \Xi^0\right\|_{1,p,1} &\le \left\|\Xi^\nu - \Xi^0\right\|_{1,p,1} + \frac{1}{\varepsilon^2}\left\|g_\nu^*\Xi^\nu - \Xi^\nu\right\|_{1,p,\varepsilon} \le c\varepsilon_\nu^{1-1/p}.\end{aligned}$$

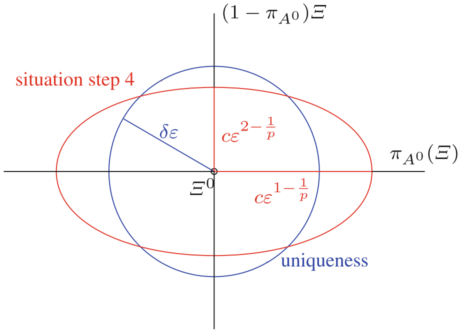
Thus, we concluded the proof of the fourth step. □

We still denote the new sequence $g_\nu^*\Xi^\nu$ by Ξ^ν in order to simplify the notation.

Step 5. There are three positive constants $\delta_1, \varepsilon_0, c$ such that for any positive $\varepsilon_\nu < \varepsilon_0$

$$\|\pi_{A^0}(A^\nu - A^0)\|_{L^2} + \|\pi_{A^0}(A^\nu - A^0)\|_{L^\infty} \le c\varepsilon_\nu^{1+\delta_1}. \tag{158}$$

Fig. 2 Uniqueness (*circle*) and the result of step 4 (*ellipse*)



End of the proof Since our sequence satisfies the assumptions of the uniqueness Theorem 9 because by the fourth step $d^{*_\varepsilon}_{\Xi^0}(\Xi^\nu - \Xi^0) = 0$ and by the fourth and the last step

$$\|\Xi^\nu - \Xi^0\|_{1,2,\varepsilon} + \|\Xi^\nu - \Xi^0\|_{\infty,\varepsilon} \le \delta\varepsilon_\nu,$$

for ν big enough $\Xi^\nu = \mathcal{T}^{\varepsilon_\nu,b}(\Xi^0)$ which is a contradiction (Fig. 2). □

Proof (*Step* 5) In order to estimate the norms of $\pi_{A^0}(A^\nu - A^0)$ we use the estimate (41), i.e.

$$\begin{aligned}
&\|\pi_A(A^\nu - A^0)\|_{L^2} + \|\nabla_t \pi_A(A^\nu - A^0)\|_{L^2}\\
&\quad \le c\|\pi_A\big(\mathcal{D}_1^{\varepsilon_\nu}(\Xi^0)(A^\nu - A^0, \Psi^\nu - \Psi^0) + *[(A^\nu - A^0)\wedge *\omega]\big)\|_{L^2}\\
&\qquad + c\|A^\nu - A^0 - \pi_A(A^\nu - A^0)\|_{L^2}\\
&\qquad + c\|\nabla_t(A^\nu - A^0 - \pi_A(A^\nu - A^0))\|_{L^2}\\
&\qquad + c\varepsilon^2\|\nabla_t(\Psi^\nu - \Psi^0)\|_{L^2} + \varepsilon^2\|\Psi^\nu - \Psi^0\|_{L^2}\\
&\qquad + c\varepsilon_\nu^2\|\mathcal{D}_2^{\varepsilon_\nu}(\Xi^0)(A^\nu - A^0, \Psi^\nu - \Psi^0)\|_{L^2}
\end{aligned}$$

which, by (82), can be written as

$$\begin{aligned}
&\|\pi_A(A^\nu - A^0)\|_{L^2} + \|\nabla_t \pi_A(A^\nu - A^0)\|_{L^2}\\
&\quad \le c\|\pi_A\big(\mathcal{D}_1^{\varepsilon_\nu}(\Xi^0 + \varepsilon_\nu^2\alpha_0)(A^\nu - A^0, \Psi^\nu - \Psi^0)\big)\|_{L^2}\\
&\qquad + c\|(A^\nu - A^0) - \pi_A(A^\nu - A^0)\|_{1,2,\varepsilon_\nu} + c\varepsilon_\nu^{3-\frac{1}{p}}\\
&\qquad + c\|\nabla_t((A^\nu - A^0) - \pi_{A^0}(A^\nu - A^0))\|_{L^2}\\
&\qquad + \varepsilon_\nu^2\|\mathcal{D}_2^{\varepsilon_\nu}(\Xi^0 + \varepsilon_\nu^2\alpha_0)(A^\nu - A^0, \Psi^\nu - \Psi^0)\|_{L^2}
\end{aligned} \tag{159}$$

where $\alpha_0 \in \operatorname{im} d^*_{A^0}$ is defined in Lemma 6 choosing $\varepsilon = 1$ and satisfies

$$\|\alpha_0\|_{2,2,1} + \|\alpha_0\|_{L^\infty} \le c; \tag{160}$$

we denote $\Xi^{1,\nu} = \Xi^0 + \varepsilon_\nu^2\alpha_0 = A^{1,\nu} + \Psi^0 dt$ and we recall also that, always by Lemma 6,

$$\|\mathcal{F}_1^\varepsilon(\Xi^{1,\nu})\|_{L^2} \le c\varepsilon^2, \quad \|\mathcal{F}_2^\varepsilon(\Xi^{1,\nu})\|_{L^2} \le c.$$

In the following, we will work with the difference $\Xi^\nu - \Xi^{1,\nu} = \tilde{\alpha}^\nu + \tilde{\psi}^\nu dt + \tilde{\phi}^\nu ds$ which by step 4 and (160) satisfies

$$\|\Xi^\nu - \Xi^{1,\nu}\|_{1,2,1} + \varepsilon_\nu^{1/p}\|\Xi^\nu - \Xi^{1,\nu}\|_{L^\infty} \le c\varepsilon_\nu^{1-1/p}. \tag{161}$$

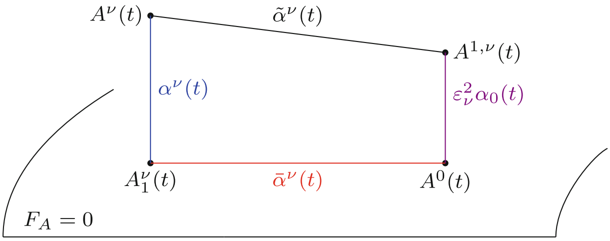


Fig. 3 The splitting of the fifth step

Furthermore we consider the decomposition

$$A^\nu - A^{1,\nu} = (A^\nu - A_1^\nu) + (A_1^\nu - A^0) + (A^0 - A^{1,\nu}) = \alpha^\nu + \bar{\alpha}^\nu - \varepsilon_\nu^2\alpha_0 = \tilde{\alpha}^\nu \tag{162}$$

where $\alpha^\nu = A^\nu - A_1^\nu$ is the 1-form defined in the first step and $\bar{\alpha}^\nu := A_1^\nu - A^0$. The idea of the proof is to use the situation described in the Fig. 3 and in order to compute the norms of $A^\nu - A^0$ we use the properties of the orthogonal splitting $H^1_{A^0} \oplus \mathrm{im}\, d_{A^0} \oplus \mathrm{im}\, d^*_{A^0}$ combined with the facts that $\alpha^\nu \in \mathrm{im}\, d^*_{A_1^\nu}$ and that the norm of $\Pi_{\mathrm{im}\, d^*_{A^0}}(\bar{\alpha}^\nu)$ can be estimate using the identity $d_{A^0}\bar{\alpha}^\nu = -\frac{1}{2}[\bar{\alpha}^\nu \wedge \bar{\alpha}^\nu]$ which can be deduced from the flat curvatures $F_{A_1^\nu}$ and F_{A^0}.

Claim 1 $\|\tilde{\alpha}^\nu - \pi_{A^0}(\tilde{\alpha}^\nu)\|_{1,2,\varepsilon_\nu} \leq c\varepsilon_\nu^{2-2/p}$.

Proof By the triangular inequality and $d^*_{A^0}\alpha_0 = 0$ we obtain

$$\|d_{A^0}(A^\nu - A^{1,\nu})\|_{L^2} \leq \varepsilon_\nu^2\|d_{A^0}\alpha_0\|_{L^2} + \|d_{A^0}(A^\nu - A^0)\|_{L^2} \leq c\varepsilon_\nu^{2-3/p}, \tag{163}$$

$$\|d^*_{A^0}\tilde{\alpha}^\nu\|_{L^2} \leq \|d^*_{A^0}(A^\nu - A^0)\|_{L^2} + \varepsilon^2\|d^*_{A^0}\alpha_0\|_{L^2} \leq c\varepsilon_\nu^{3-1/p}. \tag{164}$$

and therefore by (161), (163) and (164)

$$\|\tilde{\alpha}^\nu - \pi_{A^0}(\tilde{\alpha}^\nu)\|_{1,2,\varepsilon_\nu} \leq c\varepsilon_\nu^{2-2/p}. \tag{165}$$

□

Claim 2 $\varepsilon_\nu^2\|\mathcal{D}_2^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu)\|_{L^2} \leq c\varepsilon^{2-3/p}$.

Proof The estimate follows from

$$\mathcal{D}_2^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu) = -C_2^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu) - \mathcal{F}_2^{\varepsilon_\nu}(\Xi^{\nu,1}),$$

where $\|\mathcal{F}_2^{\varepsilon_\nu}(\Xi^{\nu,1})\|_{L^2} \leq c$ and the quadratic estimates of the Lemma 5. □

Claim 3

$$\|\pi_{A^0}(\mathcal{D}_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu)\|_{L^2} \leq c\varepsilon^{2-3/p} + \frac{1}{\varepsilon^2}\|\pi_{A^0}([\tilde{\alpha}^\nu \wedge *d_{A^0}(\tilde{\alpha}^\nu - \bar{\alpha}^\nu)])\|_{L^2}$$
$$+ \varepsilon_\nu^{1/2-2/p}\|\pi_{A^0}\tilde{\alpha}^\nu\|_{L^2}. \tag{166}$$

Proof By $\|\mathcal{F}_1^{\varepsilon_\nu}(\Xi^{\nu,1})\|_{L^2} \leq c\varepsilon_\nu^2$ and by the identity

$$\mathcal{D}_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu) = -C_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu) - \mathcal{F}_1^{\varepsilon_\nu}(\Xi^{\nu,1}),$$

we have

$$\begin{aligned}\|\pi_{A^0}(\mathcal{D}_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu,\tilde{\psi}^\nu)\|_{L^2} &\le \|\mathcal{F}_1^{\varepsilon_\nu}(\Xi^{1,\nu})\|_{L^2}+\|\pi_{A^0}(C_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu,\tilde{\psi}^\nu))\|_{L^2}\\ &\le c\varepsilon^{2-3/p}+\frac{1}{\varepsilon^2}\|\pi_{A^0}([\tilde{\alpha}^\nu\wedge *(d_{A^0}\tilde{\alpha}^\nu+\frac{1}{2}[\tilde{\alpha}^\nu\wedge\tilde{\alpha}^\nu])])\|_{L^2}\\ &\le c\varepsilon^{2-3/p}+\frac{1}{\varepsilon^2}\|\pi_{A^0}([\tilde{\alpha}^\nu\wedge *d_{A^0}(\tilde{\alpha}^\nu-\bar{\alpha}^\nu)])\|_{L^2}\\ &\quad+\varepsilon_\nu^{1/2-2/p}\|\pi_{A^0}\tilde{\alpha}^\nu\|_{L^2}\end{aligned}\tag{167}$$

where for the second inequality we estimate every term of $C_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu,\tilde{\psi}^\nu)$ using the formula (65) and for the third one we we applied

$$0=F_{A^0+\bar{\alpha}^\nu}=d_{A^0}\bar{\alpha}^\nu+\frac{1}{2}[\bar{\alpha}^\nu\wedge\bar{\alpha}^\nu],\quad \|\alpha^\nu\|_{L^2(\Sigma)}+\|d_{A^\nu}\alpha^\nu\|_{L^2(\Sigma)}\le c\varepsilon^{2-1/p}$$

and the decomposition of $\tilde{\alpha}^\nu$. □

Claim 4 $\|\nabla_t(\bar{\alpha}^\nu-\pi_{A^0}(\bar{\alpha}^\nu))\|_{L^2}\le c\varepsilon_\nu^{2-3/p}$.

Proof We denote by $\Pi_{\mathrm{im}\, d_{A^0}}$ and $\Pi_{\mathrm{im}\, d_{A^0}^*}$ respectively the projections on the linear spaces $\mathrm{im}\, d_{A^0}$ and $\mathrm{im}\, d_{A^0}^*$ using the orthogonal splitting (8). For $\bar{\alpha}^\nu-\pi_{A^0}\bar{\alpha}^\nu=d_{A^0}\bar{\gamma}^\nu+d_{A^0}\omega^\nu$, where γ is a 0-form and ω a 2-form, we then have that

$$\begin{aligned}\|\nabla_t(\bar{\alpha}^\nu-\pi_{A^0}(\bar{\alpha}^\nu))\|_{L^2} &\le c\|\bar{\alpha}^\nu-\pi_{A^0}(\bar{\alpha}^\nu)\|_{L^2}+\left\|\Pi_{\mathrm{im}\, d_{A^0}}\left(\nabla_t d_{A^0}\bar{\gamma}^\nu\right)\right\|_{L^2}\\ &\quad+\left\|\Pi_{\mathrm{im}\, d_{A^0}^*}\left(\nabla_t d_{A^0}^*\omega^\nu\right)\right\|_{L^2}\\ &\le c\varepsilon_\nu^{2-3/p}\end{aligned}$$

where the last estimate follows from the next two:

$$\begin{aligned}\left\|\Pi_{\mathrm{im}\, d_{A^0}^*}\left(\nabla_t d_{A^0}^*\omega^\nu\right)\right\|_{L^2} &\le \left\|d_{A^0}\nabla_t d_{A^0}^*\omega^\nu\right\|_{L^2}\\ &\le \left\|\nabla_t d_{A^0}\bar{\alpha}^\nu\right\|_{L^2}+\|\partial_t A^0-d_{A^0}\Psi^0\|_{L^\infty}\|\bar{\alpha}^\nu-\pi_{A^0}\bar{\alpha}^\nu\|_{L^2}\\ &\le \left\|\nabla_t[\bar{\alpha}^\nu\wedge\bar{\alpha}^\nu]\right\|_{L^2}+c\|\bar{\alpha}^\nu-\pi_{A^0}\bar{\alpha}^\nu\|_{L^2}\\ &\le c\|\bar{\alpha}^\nu\|_{L^\infty}\|\nabla_t\bar{\alpha}^\nu\|_{L^2}+c\|\bar{\alpha}^\nu-\pi_{A^0}\bar{\alpha}^\nu\|_{L^2}\le c\varepsilon_\nu^{2-3/p},\\ \left\|\Pi_{\mathrm{im}\, d_{A^0}}\left(\nabla_t d_{A^0}\bar{\gamma}^\nu\right)\right\|_{L^2} &\le \left\|\Pi_{\mathrm{im}\, d_{A^0}}\left(\nabla_t\left(\Pi_{\mathrm{im}\, d_{A^0}}\left(\tilde{\alpha}^\nu\right)\right)\right)\right\|_{L^2}\\ &\quad+\left\|\Pi_{\mathrm{im}\, d_{A^0}}\left(\nabla_t\left(\Pi_{\mathrm{im}\, d_{A^0}}\left(\alpha^\nu\right)\right)\right)\right\|_{L^2}\\ &\le c\left\|d_{A^0}^*\nabla_t\left(\Pi_{\mathrm{im}\, d_{A^0}}\left(\tilde{\alpha}^\nu\right)\right)\right\|_{L^2}\\ &\quad+\left\|\Pi_{\mathrm{im}\, d_{A^0}}\left(\nabla_t\left(\Pi_{\mathrm{im}\, d_{A^0}}\left(*[\tilde{\alpha}^\nu,\gamma^\nu]\right)\right)\right)\right\|_{L^2}\\ &\le \left\|\nabla_t d_{A^0}^*\tilde{\alpha}^\nu\right\|_{L^2}+\|\partial_t A^0-d_{A^0}\Psi^0\|_{L^\infty}\left\|\Pi_{\mathrm{im}\, d_{A^0}}\left(\tilde{\alpha}^\nu\right)\right\|\\ &\quad+c\left\|\left[\tilde{\alpha}^\nu,\gamma^\nu\right]\right\|_{L^2}+\left\|\nabla_t\left[\tilde{\alpha}^\nu,\gamma^\nu\right]\right\|_{L^2}\\ &\le c\varepsilon_\nu^{2-2/p}.\end{aligned}$$

□

Claim 5 $\varepsilon_\nu\|\nabla_t(\tilde{\alpha}^\nu - \pi_{A^0}\alpha^\nu - \bar{\alpha}^\nu)\|_{0,2,\varepsilon_\nu} + \|d_{A^0}(\tilde{\alpha}^\nu - \bar{\alpha}^\nu)\|_{0,2,\varepsilon_\nu} \le c\varepsilon_\nu^{3-6/p}$. *Therefore, using* (159) *and* (160)*, we can estimate the norm of the harmonic part by*

$$\begin{aligned}
&\|\pi_A(\tilde{\alpha}^\varepsilon)\|_{L^2} + \|\nabla_t\pi_A(\tilde{\alpha}^\varepsilon)\|_{L^2}\\
&\le \|\pi_{A^0}(\mathcal{D}_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu)\|_{L^2} + \|\tilde{\alpha}^\nu - \pi_{A^0}(\tilde{\alpha}^\nu)\|_{1,2,\varepsilon}\\
&\quad +\|\nabla_t(\tilde{\alpha}^\nu - \pi_{A^0}(\tilde{\alpha}^\nu))\|_{L^2} + \varepsilon_\nu^2\|\mathcal{D}_2^{\varepsilon_\nu}(\Xi^{1,\nu})(\tilde{\alpha}^\nu, \tilde{\psi}^\nu)\|_{L^2}
\end{aligned}$$

by the first thee claims

$$\begin{aligned}
&\le c\varepsilon_\nu^{2-3/p} + \frac{1}{\varepsilon_\nu^2}\|\pi_{A^0}([\tilde{\alpha}^\nu \wedge *d_{A^0}(\tilde{\alpha}^\nu - \bar{\alpha}^\nu)])\|_{L^2}\\
&\quad +\|\nabla_t(\tilde{\alpha}^\nu - \pi_{A^0}(\tilde{\alpha}^\nu))\|_{L^2} + \varepsilon_\nu^{1/2-2/p}\|\pi_{A^0}\tilde{\alpha}^\nu\|_{L^2}\\
&\le c\varepsilon_\nu^{2-5/p} + \frac{1}{\varepsilon_\nu^2}\|\tilde{\alpha}^\nu\|_{L^\infty}\|d_{A^0}(\tilde{\alpha}^\nu - \bar{\alpha}^\nu)\|_{L^2}\\
&\quad +\|\nabla_t(\tilde{\alpha}^\nu - \pi_{A^0}\alpha^\nu - \bar{\alpha}^\nu)\|_{L^2}\\
&\quad +\|\nabla_t(\bar{\alpha}^\nu - \pi_{A^0}\bar{\alpha}^\nu)\|_{L^2} + \varepsilon_\nu^{1/2-2/p}\|\pi_{A^0}\tilde{\alpha}^\nu\|_{L^2}
\end{aligned}$$

and because of the fourth and of the fifth claim we can conclude

$$\le c\varepsilon_\nu^{2-6/p} + \varepsilon_\nu^{1/2-2/p}\|\pi_{A^0}\tilde{\alpha}^\nu\|_{L^2}.$$

We finish therefore the proof of the fifth step by choosing $p > 6$. □

Proof (*Claim* 5) We choose an operator as follows.

$$\begin{aligned}
Q^{\varepsilon_\nu}(\Xi^0)\left(\tilde{\alpha}^\nu, \tilde{\psi}^\nu\right) &:= \mathcal{D}^{\varepsilon_\nu}(\Xi^0)\left(\tilde{\alpha}^\nu, \tilde{\psi}^\nu\right) + \frac{1}{2\varepsilon_\nu{}^2}d_{A^0}^*[\bar{\alpha}^\nu \wedge \bar{\alpha}^\nu]\\
&\quad + *\frac{1}{\varepsilon_\nu^2}\left[\tilde{\alpha}^\nu \wedge \left(d_{A^0}\tilde{\alpha}^\nu + \frac{1}{2}\left[\bar{\alpha}^\nu \wedge \bar{\alpha}^\nu\right]\right)\right]
\end{aligned}$$

Since $d_{A^0}\bar{\alpha}^\nu + \frac{1}{2}[\bar{\alpha}^\nu \wedge \bar{\alpha}^\nu] = 0$,

$$d_{A^0}^*d_{A^0}\alpha^\nu + \frac{1}{2}d_{A^0}^*[\bar{\alpha}^\nu \wedge \bar{\alpha}^\nu] = d_{A^0}^*d_{A^0}(\alpha^\nu - \bar{\alpha}^\nu) \tag{168}$$

and $\|d_{A^\nu}^*\bar{\alpha}^\nu\|_{L^2} \le c\varepsilon_\nu^{3-1/p}$ by

$$\begin{aligned}
\|d_{A^\nu}^*\bar{\alpha}^\nu\|_{L^2} &\le \|d_{A^\nu}^*(A^\nu - A^0)\|_{L^2} + \|d_{A^\nu}^*(A^\nu - A^0 - \bar{\alpha}^\nu)\|_{L^2}\\
&\le c\varepsilon_\nu^{3-1/p} + \|d_{A^\nu}^* * d_{A^\nu}\gamma^\nu\|_{L^2}\\
&\le c\varepsilon_\nu^{3-1/p} + 2\|F_{A^\nu}\|_{L^\infty}\|\gamma^\nu\|_{L^2} \le c\varepsilon_\nu^{3-1/p}
\end{aligned} \tag{169}$$

and hence

$$\|d_{A^0}^*\bar{\alpha}^\nu\|_{L^2} \le \|d_{A^\nu}^*\bar{\alpha}^\nu\|_{L^2} + c\|\alpha^\nu\|_{L^4}\|\bar{\alpha}^\nu\|_{L^4} \le c\varepsilon_\nu^{3-2/p} \tag{170}$$

$$\langle d_{A^0}d_{A^0}^*\tilde{\alpha}^\nu, \tilde{\alpha}^\nu - \bar{\alpha}^\nu\rangle \ge \|d_{A^0}^*(\tilde{\alpha}^\nu - \bar{\alpha}^\nu)\|_{L^2}^2 - c\|d_{A^0}^*\bar{\alpha}\|_{L^2}\|d_{A^0}^*(\tilde{\alpha}^\nu - \bar{\alpha}^\nu)\|_{L^2}. \tag{171}$$

Then

$$
\begin{aligned}
&\varepsilon_\nu^2\langle Q_1^{\varepsilon_\nu}(\Xi^0)\left(\tilde\alpha^\nu,\tilde\psi^\nu\right),\tilde\alpha^\nu-\pi_{A^0}\alpha^\nu-\bar\alpha^\nu\rangle\geq\|d_{A^0}^*(\tilde\alpha^\nu-\bar\alpha^\nu)\|_{L^2}^2\\
&\quad+\|d_{A^0}(\alpha^\varepsilon-\bar\alpha^\varepsilon)\|_{L^2}^2+\frac{\varepsilon_\nu^2}{2}\|\nabla_t(\tilde\alpha^\nu-\pi_{A^0}\alpha^\nu-\bar\alpha^\nu)\|_{L^2}^2\\
&\quad-c\varepsilon_\nu^{3-2/p}\|d_{A^0}^*(\tilde\alpha^\nu-\bar\alpha^\nu)\|_{L^2}\\
&\quad-c\varepsilon_\nu^2\|\tilde\alpha^\nu\|_{L^2}\|\alpha^\varepsilon-\pi_{A^0}\alpha^\nu-\bar\alpha^\nu\|_{L^2}-c\varepsilon_\nu^2\|\tilde\alpha^\nu-\pi_{A^0}\alpha^\nu-\bar\alpha^\nu\|_{0,2,\varepsilon_\nu}\|\tilde\psi^\nu\|_{0,2,\varepsilon_\nu}\\
&\quad-c\varepsilon_\nu^2|\langle\nabla_t\pi_{A^0}(\tilde\alpha^\nu),\nabla_t(\tilde\alpha^\nu-\pi_{A_0}\alpha^\nu-\bar\alpha^\nu)\rangle|\\
&\quad-c\varepsilon^2|\langle\nabla_t(\bar\alpha^\nu-\pi_{A^0}(\bar\alpha^\nu)),\nabla_t(\tilde\alpha^\nu-\bar\alpha^\nu-\pi_{A^0}\alpha^\nu)\rangle|\\
&\quad-\|\tilde\alpha^\nu\|_{L^\infty}^2\|d_{A^0}(\tilde\alpha^\nu-\bar\alpha^\nu)\|_{L^2}\|\tilde\alpha^\nu-\pi_{A^0}\alpha^\nu-\bar\alpha^\nu\|_{L^2}.
\end{aligned}\tag{172}
$$

We can conclude therefore that

$$
\begin{aligned}
&\varepsilon_\nu\|\nabla_t(\tilde\alpha^\nu-\pi_{A^0}\alpha^\nu-\bar\alpha^\nu)\|_{0,2,\varepsilon_\nu}+\|d_{A^0}(\tilde\alpha^\nu-\bar\alpha^\nu)\|_{0,2,\varepsilon_\nu}\\
&\quad\leq\varepsilon_\nu^2\|Q^{\varepsilon_\nu}(\Xi^0)\left(\tilde\alpha^\nu,\tilde\psi^\nu\right)\|_{0,2,\varepsilon_\nu}+c\varepsilon_\nu^{3-2/p}\\
&\qquad+c\varepsilon_\nu^2\|\tilde\alpha^\nu\|_{0,2,\varepsilon_\nu}+c\varepsilon_\nu^2\|\tilde\psi^\nu\|_{0,2,\varepsilon_\nu}\\
&\qquad+c\varepsilon_\nu^2\|\nabla_t\pi_{A^0}(\alpha^\nu)\|_{0,2,\varepsilon_\nu}\\
&\qquad+c\varepsilon_\nu^2\|\pi_{A^0}(\alpha^\nu)\|_{0,2,\varepsilon_\nu}\leq c\varepsilon_\nu^{3-6/p}
\end{aligned}\tag{173}
$$

where the last step follows because

$$\|Q_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\alpha^\nu,\phi^\nu)-Q_1^{\varepsilon^\nu}(\Xi^0)(\alpha^\nu,\phi^\nu)\|_{L^2}\leq c\|\alpha^\nu\|_{1,2,\varepsilon}+c\|\pi_{A^0}(\alpha^\nu)\|_{L^\infty}\|\pi_{A^0}(\alpha^\nu)\|_{L^2}$$

and

$$
\begin{aligned}
Q_1^{\varepsilon_\nu}(\Xi^{1,\nu})(\Xi^\nu-\Xi^1)=&-\mathcal{F}_1^{\varepsilon_\nu}(\Xi^{1,\nu})-C_1(\Xi^{1,\nu})(\Xi^\nu-\Xi^1)\\
&+\frac{1}{2\varepsilon_\nu^2}d_{A^0}^*[\bar\alpha^\nu\wedge\bar\alpha^\nu)]+*\frac{1}{\varepsilon_\nu^2}[\alpha^\nu,*[\bar\alpha^\nu\wedge\bar\alpha^\nu]]
\end{aligned}
$$

whose norm can be bounded by $c\varepsilon_\nu^{1-6/p}$ by the triangular and the Hölder inequalities. □

12 Proof of the main theorem

The Theorem 1 states the bijectivity of the map $\mathcal{T}^{\varepsilon,b}$ which follows directly from its Definition 1 and the Theorem 16, which prove its surjectivity, and in addition it shows that $\mathcal{T}^{b,\varepsilon}$ maps perturbed closed geodesics of Morse index k in to perturbed Yang–Mills of the same Morse index.

Theorem 18 *We choose a regular value $b>0$ of the energy E^H and an $\varepsilon_0>0$ as in Definition 1, then there is a constant $c>0$ such that for every $\Xi^0=A^0+\Psi^0dt\in\mathrm{Crit}^b_{E^H}$ the following holds. Let $\Xi^\varepsilon=A^\varepsilon+\Psi^\varepsilon dt:=\mathcal{T}^{\varepsilon,b}(\Xi^0)$, $0<\varepsilon<\varepsilon_0$, then*

1. $\varepsilon^2\langle\alpha+\psi dt,\mathcal{D}^\varepsilon(\Xi^\varepsilon)(\alpha+\psi dt)\rangle\geq c\|\alpha+\psi dt\|^2_{1,2,\varepsilon}$ *for any 1-form* $\alpha(t)+\psi(t)dt\in d_{A^0}\Omega^0(\Sigma,\mathfrak{g}_P)\oplus d^*_{A^0}\Omega^2(\Sigma,\mathfrak{g}_P)\oplus\Omega^0(\Sigma,\mathfrak{g}_P)\,dt;$
2. $\mathrm{index}_{E^H}(\Xi^0)=\mathrm{index}_{\mathcal{YM}^{\varepsilon,H}}(\Xi^\varepsilon)$.

Proof As we have already mentioned, Weber [16] proved that the Morse index of a perturbed geodesic is finite and for a generic Hamiltonian H_t its nullity is zero. We are therefore interested in the behaviour of the operator $\mathcal{D}^\varepsilon(\Xi^\varepsilon)$ respect to $\mathcal{D}^0(\Xi^0)$ and in order to investigate that we consider the two parts of the orthogonal splitting of the 1-forms

$$\Omega^1(\Sigma, \mathfrak{g}_P) = \big(d_{A^0}\Omega^0(\Sigma, \mathfrak{g}_P) \oplus d_{A^0}^*\Omega^2(\Sigma, \mathfrak{g}_P) \oplus \Omega^0(\Sigma, \mathfrak{g}_P)\, dt\big) \\ \oplus H^1_{A^0}(\Sigma, \mathfrak{g}_P).$$

We also recall that

$$\|\Xi^\varepsilon - \Xi^0\|_{2,p,\varepsilon} + \varepsilon^{\frac{1}{p}}\|\Xi^\varepsilon - \Xi^0\|_{\infty,\varepsilon} \leq c\varepsilon^2, \quad \big\|d_{A^0}(A^\varepsilon - A^0 - \alpha_0^\varepsilon)\big\|_{L^2} \leq c\varepsilon^4$$

by the Theorem 8 and the Sobolev estimate (30) provided that ε is sufficiently small where α_0^ε is defined in the Theorem 8. Integrating by parts we obtain for a

$$\begin{aligned}
&\alpha + \psi dt \in d_{A^0}\Omega^0(S^1, M, \mathfrak{g}_P) \oplus d_{A^0}^*\Omega^2(S^1, M, \mathfrak{g}_P) \oplus \Omega^0(S^1, M, \mathfrak{g}_P)\, dt :\\
&\varepsilon^2\langle \alpha + \psi dt, \mathcal{D}^\varepsilon(\Xi^\varepsilon)(\alpha + \psi dt)\rangle\\
&\quad = \varepsilon^2\langle \alpha + \psi dt, \mathcal{D}^\varepsilon(\Xi^0)(\alpha + \psi dt)\rangle\\
&\qquad + \varepsilon^2\langle \alpha + \psi dt, \big(\mathcal{D}^\varepsilon(\Xi^\varepsilon) - \mathcal{D}^\varepsilon(\Xi^0)\big)(\alpha + \psi dt)\rangle\\
&\quad \geq c\|\alpha + \psi dt\|^2_{1,2,\varepsilon} + \varepsilon^2\langle \alpha + \psi dt, \big(\mathcal{D}^\varepsilon(\Xi^\varepsilon) - \mathcal{D}^\varepsilon(\Xi^0)\big)(\alpha + \psi dt)\rangle\\
&\quad \geq c\|\alpha + \psi dt\|^2_{1,2,\varepsilon} - c\varepsilon^{-\frac{1}{2}}\|\Xi^\varepsilon - \Xi^0\|_{1,2,\varepsilon}\|\alpha + \psi dt\|_{1,2,\varepsilon}\|\alpha + \psi dt\|_{0,2,\varepsilon}\\
&\quad \geq c\|\alpha + \psi dt\|^2_{1,2,\varepsilon} - c\varepsilon^{3/2}\|\alpha + \psi dt\|^2_{1,2,\varepsilon}\\
&\quad \geq c\|\alpha + \psi dt\|^2_{1,2,\varepsilon}
\end{aligned} \tag{174}$$

where the third step follows by the quadratic estimates of the Lemma 4 from the Sobolev estimate of Lemma 4 and the last one holds for ε small enough. We choose now $\alpha(t) \in H^1_{A^0}(\Sigma, \mathfrak{g}_P)$ and then we pick $\psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$, such that

$$d_{A^0}^* d_{A^0}\psi = -2 * [\alpha \wedge *(\partial_t A^0 - d_{A^0}\Psi^0)]$$

Then

$$\langle \alpha + \psi dt, \mathcal{D}^\varepsilon(\Xi^\varepsilon)(\alpha + \psi dt)\rangle = \langle \mathcal{D}^0(\Xi^0)(\alpha), \alpha\rangle + \varepsilon^2\|\nabla_t\psi\|^2_{L^2} + Q$$

where

$$\begin{aligned}
Q :=\ & \frac{1}{\varepsilon^2}\left\langle * \left[\alpha \wedge * \left(d_{A^0}(A^\varepsilon - A^0 - \alpha_0) + \frac{1}{2}\left[(A^\varepsilon - A^0) \wedge (A^\varepsilon - A^0)\right]\right)\right], \alpha\right\rangle\\
& + \frac{1}{\varepsilon^2}\left\|\left[(A^\varepsilon - A^0) \wedge \alpha\right]\right\|^2_{L^2} + \frac{1}{\varepsilon^2}\left\|\left[(A^\varepsilon - A^0) \wedge *\alpha\right]\right\|^2_{L^2}\\
& + \varepsilon^2\|[(\Psi^\varepsilon - \Psi^0), \psi]\|^2_{L^2} - \langle d * X_t(A^\varepsilon)\alpha - d * X_t(A^0)\alpha, \alpha\rangle\\
& - \big\langle 2\big[\psi, \big(\nabla_t(A^\varepsilon - A^0) - d_{A^0}(\Psi^\varepsilon - \Psi^0) - \big[(A^\varepsilon - A^0) \wedge (\Psi^\varepsilon - \Psi^0)\big]\big)\big], \alpha\big\rangle\\
& - \big\langle 2 * \big[\alpha \wedge * \big(\nabla_t(A^\varepsilon - A^0) - d_{A^0}(\Psi^\varepsilon - \Psi^0)\big)\big], \psi\big\rangle\\
& + \big\langle 2 * \big[\alpha \wedge * \big[(A^\varepsilon - A^0) \wedge (\Psi^\varepsilon - \Psi^0)\big]\big], \psi\big\rangle\\
& + \|[(A^\varepsilon - A^0), \psi]\|^2_{L^2} + \|[(\Psi^\varepsilon - \Psi^0) \wedge \alpha]\|^2_{L^2}
\end{aligned}$$

and hence

$$|Q| \leq c_1\varepsilon^{1/2}\left(\|\alpha\|^2_{L^2} + \|\nabla_t\alpha\|^2_{L^2}\right) \tag{175}$$

for a positive constant c_1; in order to compute (175) we need also to use

$$\|\psi\|_{L^2} \leq \|\alpha\|_{L^2}, \quad \|\alpha\|_{L^4} \leq \|\alpha\|_{L^2} + \|\nabla_t\alpha\|_{L^2}$$

where the first estimate follows from the definition of ψ and the second from the Sobolev inequality. Therefore there is a constant $c > 0$ such that if α is an element of the negative eigenspace of $\mathcal{D}^0(\Xi^0)$, then

$$\begin{aligned}&\langle \alpha + \psi dt, \mathcal{D}^\varepsilon(\Xi^\varepsilon)(\alpha + \psi dt)\rangle \\ &\quad \leq -c\left(\|\alpha\|_{L^2} + \|\nabla_t\alpha\|_{L^2}\right)^2 + c_1\varepsilon^{1/2}\left(\|\alpha\|^2_{L^2} + \|\nabla_t\alpha\|^2_{L^2}\right); \end{aligned} \tag{176}$$

and if α is in the positive eigenspace for $\mathcal{D}^0(\Xi^0)$, then

$$\begin{aligned}&\langle \alpha + \psi dt, \mathcal{D}^\varepsilon(\Xi^\varepsilon)(\alpha + \psi dt)\rangle \\ &\quad \geq c\left(\|\alpha\|_{L^2} + \|\nabla_t\alpha\|_{L^2}\right)^2 - c_1\varepsilon^{1/2}\left(\|\alpha\|^2_{L^2} + \|\nabla_t\alpha\|^2_{L^2}\right). \end{aligned} \tag{177}$$

Thus, by (174), (176) and (177) the dimensions of the negative eigenspaces of $\mathcal{D}^0(\Xi^0)$ and $\mathcal{D}^\varepsilon(\Xi^\varepsilon)$ are equal provided that ε is small enough and hence we can conclude that the Morse indices are equal. □

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Appendix A: Estimates on the surface

The first two lemmas were proved in [3] (lemma 7.6 and lemma 8.2) for $p > 2$ and $q = \infty$; the proofs in the case $p = 2$ and $2 \leq q < \infty$ is similar.

Lemma 11 *We choose $p > 2$ and $q = \infty$ or $p = 2$ and $2 \leq q < \infty$. Then there exist two positive constants δ and c such that for every connection $A \in \mathcal{A}(P)$ with*

$$\|F_A\|_{L^p(\Sigma)} \leq \delta$$

there are estimates

$$\|\psi\|_{L^q(\Sigma)} \leq c\|d_A\psi\|_{L^p(\Sigma)}, \qquad \|d_A\psi\|_{L^q(\Sigma)} \leq c\|d_A * d_A\psi\|_{L^p(\Sigma)},$$

for $\psi \in \Omega^0(\Sigma, \mathfrak{g}_P)$.

Lemma 12 *We choose $p > 2$ and $q = \infty$ or $p = 2$ and $2 \leq q < \infty$. Then there exist two positive constants δ and c such that the following holds. For every connection $A \in \mathcal{A}(P)$ with*

$$\|F_A\|_{L^p(\Sigma)} \leq \delta$$

there exists a unique section $\eta \in \Omega^0(\Sigma, \mathfrak{g}_P)$ such that

$$F_{A+*d_A\eta} = 0, \qquad \|d_A\eta\|_{L^q(\Sigma)} \leq c\|F_A\|_{L^p(\Sigma)}.$$

The following lemma is a simplified version of the lemma B.2. in [13] where Salamon allows also to modify the complex structure on Σ if it is C^1-closed to a fixed one.

Lemma 13 *Fix a connection $A^0 \in \mathcal{A}_0(P)$. Then, for every $\delta > 0$, $C > 0$, and $p \geq 2$, there exists a constant $c = c(\delta, C, A^0) \geq 1$ such that, if $A \in \mathcal{A}(P)$ satisfy $\|A - A^0\|_{L^\infty(\Sigma)} \leq C$ then, for every $\psi \in \Omega^0(\Sigma, \mathfrak{g}_P)$ and every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,*

$$\|\psi\|^p_{L^p(\Sigma)} \leq \delta \|d_A \psi\|^p_{L^p(\Sigma)} + c\|\psi\|^p_{L^2(\Sigma)}, \tag{178}$$

$$\|\alpha\|^p_{L^p(\Sigma)} \leq \delta \left(\|d_A \alpha\|^p_{L^p(\Sigma)} + \|d_A * \alpha\|^p_{L^p(\Sigma)}\right) + c\|\alpha\|^p_{L^2(\Sigma)}. \tag{179}$$

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